

Nikishin systems on star-like sets: algebraic properties and weak asymptotics of the associated multiple orthogonal polynomials

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Abstract

Polynomials $Q_n(z)$, $n = 0, 1, \dots$, that are multi-orthogonal with respect to a Nikishin system of $p \geq 1$ compactly supported measures over the star-like set of $p + 1$ rays $S_+ := \{z \in \mathbb{C} : z^{p+1} \geq 0\}$ are investigated. We prove that the Nikishin system is normal, that the polynomials satisfy a three-term recurrence relation of order $p + 1$ of the form $zQ_n(z) = Q_{n+1}(z) + a_n Q_{n-p}(z)$ with $a_n > 0$ for all $n \geq p$, and that the nonzero roots of Q_n are all simple and located in S_+ . Under the assumption of regularity (in the sense of Stahl and Totik) of the measures generating the Nikishin system, we describe the asymptotic zero distribution and weak behavior of the polynomials Q_n in terms of a vector equilibrium problem for logarithmic potentials. Under the same regularity assumptions, a theorem on the convergence of the Hermite-Padé approximants to the Nikishin system of Cauchy transforms is proven.

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1 Introduction

This work is motivated by the studies [2, 1, 3] on sequences of polynomials $\{Q_n\}_{n=0}^\infty$ satisfying a recurrence relation of the form

$$zQ_n(z) = Q_{n+1}(z) + a_n Q_{n-p}(z), \quad a_n > 0, \quad n \geq p, \quad (1.1)$$

where p is a fixed positive integer.

Some well-known families of polynomials satisfy this type of recurrence relation with the coefficients a_n all being equal to some constant a . For instance, when $p = 1$ and $a_n = 1$ for all $n \geq 1$, the polynomials Q_n resulting from the pairs of initial conditions $Q_0(z) = 2$, $Q_1(z) = z$, and $Q_0(z) = 1$, $Q_1(z) = z$, are, respectively, the Chebyshev polynomials of the first and second kind for the interval $[-2, 2]$. As a way of generalizing the Chebyshev polynomials of the first kind, one can set in (1.1) $a_n = 1/p$, $n \geq p$, and $Q_0(z) = p + 1$, $Q_\ell = z^\ell$, $\ell = 1, \dots, p$, which generates the sequence of Faber polynomials associated with a hypocycloid of $p + 1$ cusps. Many interesting properties of these Faber polynomials were established in [8]. For instance, their zeros are all located in the star-like set of $p + 1$ rays

$$S_+ := \{z \in \mathbb{C} : z^{p+1} \geq 0\},$$

more precisely, they are contained, interlace, and form a dense subset of $\{z \in S_+ : |z| < (p + 1)/p\}$.

It was proven in [2] that with the initial conditions

$$Q_\ell(z) = z^\ell, \quad 0 \leq \ell \leq p, \quad (1.2)$$

the polynomials generated by (1.1) are in fact multi-orthogonal (in the same non-Hermitian sense of Definition 2.3 below) with respect to a system of p complex measures μ_1, \dots, μ_p supported on S_+ . These measures can be viewed as spectral measures [2, 1] of the difference operator given in the standard basis of the Hilbert space $l^2(\mathbb{N})$ by the infinite $(p + 2)$ -banded Hessenberg matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_p & 0 & 0 & 0 & 0 & \dots & \dots \\ 0 & a_{p+1} & 0 & 0 & 0 & \dots & \dots \\ 0 & 0 & a_{p+2} & 0 & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \quad (1.3)$$

Their Cauchy transforms

$$\int_{S_+} \frac{d\mu_j(t)}{z-t}, \quad 1 \leq j \leq p,$$

are the resolvent or Weyl functions of the operator. We remark that the spectral measures are of the form $d\mu_j(t) = t^{1-j} d\nu_j(t^{p+1})$, $j = 1, \dots, p$, where ν_j is a positive measure supported on $S_+^{p+1} = \mathbb{R}_+$. Hence μ_1 is rotationally invariant, and the rest are rotationally invariant up to a monomial factor.

The l^1 perturbation of the constant coefficient case

$$\sum_{n=p}^{\infty} |a_n - a| < \infty, \quad a > 0, \quad (1.4)$$

was investigated in [1]. Here, the strong asymptotics of the polynomials Q_n determined by (1.1), (1.2), and (1.4), as well as properties of the measures μ_j , were derived. For instance, it was proven that these spectral measures are absolutely continuous, and a formal connection of these measures with a Nikishin-type system was obtained. In [3], this connection was explicitly established in the case of periodic recurrence coefficients (see Section 2.4 of [3]), and many algebraic and asymptotic properties of the Riemann-Hilbert minors associated with the polynomials Q_n satisfying (1.1)–(1.2) were given.

Motivated by these results, we investigate in this paper polynomials Q_n that are multi-orthogonal with respect to a Nikishin system of p measures (defined in analogy to the classical sense) supported over the star-like set S_+ . As we will see in Section 3 below, such Q_n 's happen to satisfy the recurrence relation (1.1). Our goal is to understand how the properties of the measures generating the Nikishin system affect the multi-orthogonal polynomials Q_n and the recurrence coefficients a_n , in particular, what their asymptotic behavior is as $n \rightarrow \infty$. Thus, in the context of inverse spectral problems, our investigation sheds some light into the properties of the operator (1.3).

Nikishin systems of functions (the Cauchy transforms of a Nikishin system of measures on intervals of the real line) were first introduced in [10] as the first wide class of functions possessing convergent Hermite-Padé approximants. While in his original paper [10] Nikishin proved this convergence only for a system of two measures and diagonal multi-indices, great progress has been made since then for any number of intervals and arbitrary multi-indices (see, for instance, [5]). Our work can also be viewed within the context of rational approximation as a generalization of Nikishin systems from real intervals to star-like sets.

The content of the paper is organized in five sections. Sections 2 and 3 are, for the most part, of an algebraic nature, and they have been linearly structured so as to have any result needed for a given topic stated and proven beforehand. The Nikishin system and other related hierarchies of measures, together with the multi-orthogonal polynomials and their associated functions of the second kind, are introduced in Section 2. Among the many relations and properties proven in that section figure the normality of the Nikishin system and the location of the zeros of the multi-orthogonal polynomials and of the functions of the second kind. In Section 3, we prove the recurrence relation (1.1) for the multi-orthogonal polynomials, including the (nontrivial) positivity of the recurrence coefficients. In Section 5, we describe the asymptotic zero distribution and weak behavior of the polynomials Q_n in terms of a vector equilibrium problem for logarithmic potentials, under the assumption that the measures generating the Nikishin system are regular in the sense of Stahl and Totik. A weak convergence theorem for the coefficients of the recurrence relation is also obtained. Finally, in Section 6, and under the same regularity assumptions, a theorem on the convergence of the Hermite-Padé approximants to the Nikishin system of Cauchy transforms is proven.

For a first, basic understanding of the statements of the main results of the paper, we recommend reading just the following parts:

- Subsections 2.1 and 2.2 (for a more streamlined reading, Propositions 2.1 and 2.2 can be skipped).

- Propositions 2.16 and 2.19.
- Propositions 3.1 and 3.2, Theorem 3.5.
- Subsections 5.1 and 6.1.

Many of the results in this paper were already obtained in [9] for Nikishin systems of $p = 2$ measures on a star-like set of three rays. Ratio asymptotics for the multiple orthogonal polynomials and the limiting behavior of the recurrence coefficients were also established in [9] for $p = 2$ under a Rakhmanov-type condition on the Nikishin system of measures. The extension of these results to any value of p will be accomplished in a subsequent work.

2 Nikishin systems on stars

2.1 Definition and basic properties of the Nikishin system

Let $p \geq 1$ be an integer, and let

$$S := \{z \in \mathbb{C} : z^{p+1} \in \mathbb{R}\}.$$

Then

$$S_- = e^{\frac{\pi i}{p+1}} S_+.$$

We construct p finite stars contained in S as follows:

$$\Gamma_j := \{z \in \mathbb{C} : z^{p+1} \in [a_j, b_j]\}, \quad 0 \leq j \leq p-1,$$

where

$$\begin{aligned} 0 \leq a_j < b_j < \infty, \quad j \equiv 0 \pmod{2}, \\ -\infty < a_j < b_j \leq 0, \quad j \equiv 1 \pmod{2}, \end{aligned}$$

so that $\Gamma_j \subset S_+$ if j is even, and $\Gamma_j \subset S_-$ if j is odd. We assume throughout that $\Gamma_j \cap \Gamma_{j+1} = \emptyset$ for all $0 \leq j \leq p-2$, that is, any two consecutive stars do not meet at the origin.

We define now a Nikishin system on $(\Gamma_0, \dots, \Gamma_{p-1})$. For each $0 \leq j \leq p-1$, let σ_j denote a positive, rotationally invariant measure on Γ_j , with infinitely many points in its support. These will be the measures generating the Nikishin system.

Let

$$\widehat{\mu}(x) := \int \frac{d\mu(t)}{x-t}$$

denote the Cauchy transform of a complex measure μ , and let μ_1, \dots, μ_N be $N \geq 1$ measures such that μ_j and μ_{j+1} have disjoint supports for every $1 \leq j \leq N-1$. We define the measure $\langle \mu_1, \dots, \mu_N \rangle$ by the following recursive procedure. For $N = 1$, $\langle \mu_1 \rangle := \mu_1$, for $N = 2$,

$$d\langle \mu_1, \mu_2 \rangle(x) := \widehat{\mu}_2(x) d\mu_1(x),$$

and for $N > 2$,

$$\langle \mu_1, \dots, \mu_N \rangle := \langle \mu_1, \langle \mu_2, \dots, \mu_N \rangle \rangle.$$

We then define the Nikishin system $(s_0, \dots, s_{p-1}) = \mathcal{N}(\sigma_0, \dots, \sigma_{p-1})$ generated by the vector of p measures $(\sigma_0, \dots, \sigma_{p-1})$ by setting

$$s_j := \langle \sigma_0, \dots, \sigma_j \rangle, \quad 0 \leq j \leq p-1. \quad (2.1)$$

Notice that these measures s_j are supported on the first star Γ_0 .

It is convenient, however, to think of this Nikishin system as the first row of the following hierarchy of measures $s_{k,j}$,

$$\begin{array}{cccccc}
s_{0,0} & s_{0,1} & s_{0,2} & \cdots & s_{0,p-1} & \\
& s_{1,1} & s_{1,2} & \cdots & s_{1,p-1} & \\
& & s_{2,2} & \cdots & s_{2,p-1} & \\
& & & \ddots & \vdots & \\
& & & & s_{p-1,p-1} &
\end{array} \tag{2.2}$$

where

$$s_{k,j} = \langle \sigma_k, \dots, \sigma_j \rangle, \quad 0 \leq k \leq j \leq p-1. \tag{2.3}$$

More descriptively, the measures $s_{k,j}$ are inductively defined by setting

$$\begin{aligned}
s_{k,k} &:= \sigma_k, \quad 0 \leq k \leq p-1, \\
ds_{k,j}(z) &= \int_{\Gamma_{k+1}} \frac{ds_{k+1,j}(t)}{z-t} d\sigma_k(z), \quad 0 \leq k < j \leq p-1.
\end{aligned} \tag{2.4}$$

Notice then that for each pair k, j with $0 \leq k \leq j \leq p-1$, $(s_{k,k}, \dots, s_{k,j}) = \mathcal{N}(\sigma_k, \dots, \sigma_j)$ is the Nikishin system generated by $(\sigma_k, \dots, \sigma_j)$.

Throughout the paper we will use the notation

$$\omega := e^{\frac{2\pi i}{p+1}}.$$

The following proposition summarizes several basic properties that will be needed later.

Proposition 2.1. *For every $0 \leq k \leq j \leq p-1$, the measure $s_{k,j}$ satisfies the symmetry property*

$$ds_{k,j}(\omega z) = \omega^{k-j} ds_{k,j}(z). \tag{2.5}$$

Also, for every integrable f on Γ_k , we have

$$\int_{\Gamma_k} f(\omega z) ds_{k,j}(z) = \omega^{j-k} \int_{\Gamma_k} f(z) ds_{k,j}(z), \tag{2.6}$$

$$\int_{\Gamma_k} \overline{f(\overline{z})} ds_{k,j}(z) = \overline{\int_{\Gamma_k} f(z) ds_{k,j}(z)}. \tag{2.7}$$

Proof. The proof is by reverse induction on k . The relation (2.5) holds trivially in the case $k = p-1$. Assume that (2.5) holds for $k+1$, that is, for every j satisfying $k+1 \leq j \leq p-1$ one has $ds_{k+1,j}(\omega z) = \omega^{k+1-j} ds_{k+1,j}(z)$. Since σ_k is rotationally invariant, applying (2.4) we obtain

$$\begin{aligned}
ds_{k,j}(\omega z) &= \left(\int_{\Gamma_{k+1}} \frac{ds_{k+1,j}(t)}{\omega z - t} \right) d\sigma_k(\omega z) \\
&= \left(\int_{\Gamma_{k+1}} \frac{ds_{k+1,j}(t)}{\omega z - t} \right) d\sigma_k(z) \\
&= \omega^{-1} \left(\int_{\Gamma_{k+1}} \frac{ds_{k+1,j}(\omega t)}{z - t} \right) d\sigma_k(z) \\
&= \omega^{k-j} \left(\int_{\Gamma_{k+1}} \frac{ds_{k+1,j}(t)}{z - t} \right) d\sigma_k(z).
\end{aligned}$$

Formula (2.6) follows immediately from (2.5). Observe that the rotational invariance of the measures σ_k implies that $d\sigma_k(t) = d\sigma_k(\bar{t})$, and if we assume that for every integrable function f_{k+1} defined on Γ_{k+1} ,

$$\int_{\Gamma_{k+1}} \overline{f_{k+1}(\bar{z})} ds_{k+1,j}(z) = \overline{\int_{\Gamma_{k+1}} f_{k+1}(z) ds_{k+1,j}(z)},$$

then for every integrable f_k on Γ_k , we have

$$\begin{aligned} \int_{\Gamma_k} \overline{f_k(\bar{z})} ds_{k,j}(z) &= \int_{\Gamma_k} \left(\int_{\Gamma_{k+1}} \frac{ds_{k+1,j}(t)}{\bar{z} - t} \right) \overline{f_k(z)} d\sigma_k(z) \\ &= \int_{\Gamma_k} f_k(z) \overline{\left(\int_{\Gamma_{k+1}} \frac{ds_{k+1,j}(t)}{\bar{z} - t} \right)} d\sigma_k(z) \\ &= \int_{\Gamma_k} f_k(z) \left(\int_{\Gamma_{k+1}} \frac{ds_{k+1,j}(t)}{z - t} \right) d\sigma_k(z). \end{aligned}$$

This completes the induction and the proof. \square

For every $0 \leq j \leq p-1$, we shall denote by σ_j^* the push-forward of σ_j under the map $z \mapsto z^{p+1}$, that is, σ_j^* is the measure on $[a_j, b_j]$ such that for every Borel set $E \subset [a_j, b_j]$,

$$\sigma_j^*(E) := \sigma_j(\{z : z^{p+1} \in E\}). \quad (2.8)$$

We now construct, out of these σ_j^* , a new hierarchy of measures $\mu_{k,j}$, $0 \leq k \leq j \leq p-1$:

$$\begin{array}{ccccccc} \mu_{0,0} & \mu_{0,1} & \mu_{0,2} & \cdots & \mu_{0,p-1} & & \\ & \mu_{1,1} & \mu_{1,2} & \cdots & \mu_{1,p-1} & & \\ & & \mu_{2,2} & \cdots & \mu_{2,p-1} & & \\ & & & \ddots & \vdots & & \\ & & & & \mu_{p-1,p-1} & & \end{array} \quad (2.9)$$

where the measures $\mu_{k,j}$ are inductively defined by setting

$$\mu_{k,k} := \sigma_k^*, \quad 0 \leq k \leq p-1,$$

$$d\mu_{k,j}(\tau) = \left(\tau \int_{a_{k+1}}^{b_{k+1}} \frac{d\mu_{k+1,j}(s)}{\tau - s} \right) d\sigma_k^*(\tau), \quad \tau \in [a_k, b_k], \quad 0 \leq k < j \leq p-1. \quad (2.10)$$

In the following result we describe the relationship between the measures $\mu_{k,j}$ and $s_{k,j}$.

Proposition 2.2. *For every $0 \leq k \leq j \leq p-1$, we have*

$$\int_{\Gamma_k} \frac{ds_{k,j}(t)}{z - t} = z^{p+k-j} \int_{a_k}^{b_k} \frac{d\mu_{k,j}(\tau)}{z^{p+1} - \tau},$$

that is,

$$\widehat{s}_{k,j}(z) = z^{p+k-j} \widehat{\mu}_{k,j}(z^{p+1}). \quad (2.11)$$

Hence, for every integrable function f on $[a_k, b_k]$,

$$\int_{a_k}^{b_k} f(\tau) d\mu_{k,j}(\tau) = \int_{\Gamma_k} f(z^{p+1}) z^{j-k} ds_{k,j}(z). \quad (2.12)$$

Proof. Using (2.5) we find that

$$\int_{\Gamma_k} \frac{t^l ds_{k,j}(t)}{z - t^{p+1}} = \omega^{l+k-j} \int_{\Gamma_k} \frac{t^l ds_{k,j}(t)}{z - t^{p+1}}.$$

This implies that

$$\int_{\Gamma_k} \frac{t^l ds_{k,j}(t)}{z - t^{p+1}} = 0, \quad l \not\equiv j - k \pmod{p+1}.$$

Since $0 \leq j - k \leq j \leq p - 1$, it follows that

$$\begin{aligned} z^{p+k-j} \int_{\Gamma_k} \frac{t^{j-k} ds_{k,j}(t)}{z^{p+1} - t^{p+1}} &= \int_{\Gamma_k} \frac{ds_{k,j}(t)}{z - t} - \sum_{0 \leq l \leq p, l \neq j-k} z^{p-l} \int_{\Gamma_k} \frac{t^l ds_{k,j}(t)}{z^{p+1} - t^{p+1}} \\ &= \int_{\Gamma_k} \frac{ds_{k,j}(t)}{z - t}. \end{aligned}$$

This already proves that (2.11) holds true for $k = j$. Assume it also holds for $k + 1$. Then

$$\begin{aligned} \int_{\Gamma_k} \frac{ds_{k,j}(t)}{z - t} &= z^{p+k-j} \int_{\Gamma_k} \frac{t^{j-k} ds_{k,j}(t)}{z^{p+1} - t^{p+1}} = z^{p+k-j} \int_{\Gamma_k} \frac{t^{j-k} \widehat{s}_{k+1,j}(t) d\sigma_k(t)}{z^{p+1} - t^{p+1}} \\ &= z^{p+k-j} \int_{\Gamma_k} \frac{t^{p+1} \widehat{\mu}_{k+1,j}(t^{p+1}) d\sigma_k(t)}{z^{p+1} - t^{p+1}} \\ &= z^{p+k-j} \int_{a_k}^{b_k} \frac{d\mu_{k,j}(\tau)}{z^{p+1} - \tau}. \end{aligned}$$

This finishes the proof of (2.11).

Now, the relation (2.12) is obviously true for $j = k$, since in this case $s_{k,k} = \sigma_k$ and $\mu_{k,k} = \sigma_k^*$. If $j > k$, then using (2.11) we find

$$\begin{aligned} \int_{a_k}^{b_k} f(\tau) d\mu_{k,j}(\tau) &= \int_{a_k}^{b_k} f(\tau) \tau \widehat{\mu}_{k+1,j}(\tau) d\sigma_k^*(\tau) = \int_{\Gamma_k} f(z^{p+1}) z^{p+1} \widehat{\mu}_{k+1,j}(z^{p+1}) d\sigma_k(z) \\ &= \int_{\Gamma_k} f(z^{p+1}) z^{j-k} \widehat{s}_{k+1,j}(z) d\sigma_k(z). \end{aligned}$$

□

2.2 Multiple orthogonal polynomials and functions of the second kind

Definition 2.3. Let $\{Q_n(z)\}_{n=0}^\infty$ be the sequence of monic polynomials of lowest degree that satisfy the following non-hermitian orthogonality conditions:

$$\int_{\Gamma_0} Q_n(z) z^l ds_j(z) = 0, \quad l = 0, \dots, \left\lfloor \frac{n-j-1}{p} \right\rfloor, \quad 0 \leq j \leq p-1. \quad (2.13)$$

In more detail, (2.13) asserts that the polynomial Q_{mp+r} must satisfy the orthogonality relations

$$\int_{\Gamma_0} Q_{mp+r}(z) z^l ds_j(z) = 0, \quad l = 0, \dots, m-1, \quad 0 \leq j \leq p-1, \quad (2.14)$$

$$\int_{\Gamma_0} Q_{mp+r}(z) z^m ds_j(z) = 0, \quad 0 \leq j \leq r-1. \quad (2.15)$$

In what follows we will use the notation

$$d_n := \deg(Q_n), \quad n \geq 0.$$

Using (2.6)-(2.7), one easily sees that the polynomials $Q_n(z)$, $Q_n(\omega z)$ and $\overline{Q_n(\bar{z})}$ satisfy the same orthogonality relations (2.13). Thus, by the uniqueness of Q_n , we have that

$$Q_n(\omega z) = \omega^{d_n} Q_n(z), \quad Q_n(z) = \overline{Q_n(\bar{z})}, \quad n \geq 0. \quad (2.16)$$

Let $0 \leq \ell \leq p$ be such that $d_n \equiv \ell \pmod{p+1}$, so that

$$d_n = d(p+1) + \ell, \quad d := \left\lfloor \frac{d_n}{p+1} \right\rfloor.$$

Then, the first relation in (2.16) implies that

$$Q_n(t) = t^\ell Q_d(t^{p+1}) \quad (2.17)$$

for some polynomial Q_d of exact degree d .

The polynomials Q_n are intrinsically related to the so-called functions of the second kind, which we define next.

Definition 2.4. Set $\Psi_{n,0} = Q_n$ and let

$$\Psi_{n,k}(z) = \int_{\Gamma_{k-1}} \frac{\Psi_{n,k-1}(t)}{z-t} d\sigma_{k-1}(t), \quad k = 1, \dots, p.$$

Observe that $\Psi_{n,k}$ is analytic in $\mathbb{C} \setminus \Gamma_{k-1}$. Our next proposition shows that the function $\Psi_{n,k}$, $0 \leq k \leq p-1$, satisfies multiple orthogonality conditions similar to those satisfied by Q_n but with respect to the Nikishin system given by the k th row of the hierarchy (2.2). Note that the function $\Psi_{n,p}$ is excluded from this proposition.

Proposition 2.5. For each $k = 0, \dots, p-1$, the function $\Psi_{n,k}$ satisfies the following orthogonality conditions

$$\int_{\Gamma_k} \Psi_{n,k}(z) z^l ds_{k,j}(z) = 0, \quad 0 \leq l \leq \left\lfloor \frac{n-j-1}{p} \right\rfloor, \quad k \leq j \leq p-1. \quad (2.18)$$

Proof. The proof is by induction on k . For $k = 0$, the orthogonality conditions (2.18) coincide with (2.13), so they are valid by definition. Suppose that (2.18) holds for some k with $0 \leq k \leq p-2$. We have

$$\begin{aligned} & \int_{\Gamma_{k+1}} \Psi_{n,k+1}(z) z^l ds_{k+1,j}(z) \\ &= \int_{\Gamma_{k+1}} \left(\int_{\Gamma_k} \frac{\Psi_{n,k}(t)}{z-t} d\sigma_k(t) \right) z^l ds_{k+1,j}(z) \\ &= \int_{\Gamma_k} \int_{\Gamma_{k+1}} \frac{\Psi_{n,k}(t)(z^l - t^l + t^l)}{z-t} ds_{k+1,j}(z) d\sigma_k(t) \\ &= \int_{\Gamma_k} \Psi_{n,k}(t) p_l(t) d\sigma_k(t) - \int_{\Gamma_k} \Psi_{n,k}(t) t^l ds_{k,j}(t), \end{aligned}$$

where $p_l(t)$ denotes the polynomial

$$p_l(t) = \int_{\Gamma_{k+1}} \frac{z^l - t^l}{z-t} ds_{k+1,j}(z).$$

Therefore, it is clear that if $0 \leq l \leq \left\lfloor \frac{n-j-1}{p} \right\rfloor$, $k+1 \leq j \leq p-1$, then the last two integrals in the above chain of equalities are zero. \square

Proposition 2.6. *The functions $\Psi_{n,k}$ satisfy the symmetry property*

$$\Psi_{n,k}(\omega z) = \omega^{d_n-k} \Psi_{n,k}(z), \quad k = 0, \dots, p, \quad n \geq 0, \quad (2.19)$$

where, as above, d_n is the degree of Q_n .

Proof. The proof is again by induction on k . The case $k = 0$ is the already proved symmetry property (2.16) for the polynomials Q_n . Assuming that (2.19) holds for k , then

$$\Psi_{n,k+1}(\omega z) = \int_{\Gamma_k} \frac{\Psi_{n,k}(t)}{\omega z - t} d\sigma_k(t) = \int_{\Gamma_k} \frac{\Psi_{n,k}(\omega t)}{\omega z - \omega t} d\sigma_k(t) = \omega^{d_n-k-1} \Psi_{n,k+1}(z).$$

□

We now seek to find an analogue of the polynomial Q_d in (2.17) for the functions $\Psi_{n,k}$. To accomplish that, we first need the following representation.

Proposition 2.7. *Assume that $d_n \equiv \ell \pmod{p+1}$ with $0 \leq \ell \leq p$. Then, for each $k = 1, \dots, p$ we have*

$$\Psi_{n,k}(z) = z^{p-s} \int_{\Gamma_{k-1}} \frac{\Psi_{n,k-1}(t) t^s}{z^{p+1} - t^{p+1}} d\sigma_{k-1}(t), \quad (2.20)$$

where s is the only integer in $\{0, \dots, p\}$ such that $s \equiv k-1-\ell \pmod{p+1}$, that is,

$$s = \begin{cases} k-1-\ell, & \ell < k, \\ p+k-\ell, & k \leq \ell. \end{cases} \quad (2.21)$$

Proof. Let $1 \leq k \leq p$. By definition,

$$\Psi_{n,k}(z) = \int_{\Gamma_{k-1}} \frac{\Psi_{n,k-1}(t)}{z-t} d\sigma_{k-1}(t) = \sum_{l=0}^p z^{p-l} \int_{\Gamma_{k-1}} \frac{\Psi_{n,k-1}(t) t^l}{z^{p+1} - t^{p+1}} d\sigma_{k-1}(t).$$

Now, using (2.19) we deduce that for each $l = 0, \dots, p$,

$$\begin{aligned} \int_{\Gamma_{k-1}} \frac{\Psi_{n,k-1}(t) t^l}{z - t^{p+1}} d\sigma_{k-1}(t) &= \int_{\Gamma_{k-1}} \frac{\Psi_{n,k-1}(\omega t) (\omega t)^l}{z - (\omega t)^{p+1}} d\sigma_{k-1}(t) \\ &= \omega^{d_n-k+1+l} \int_{\Gamma_{k-1}} \frac{\Psi_{n,k-1}(t) t^l}{z - t^{p+1}} d\sigma_{k-1}(t). \end{aligned}$$

Hence,

$$\int_{\Gamma_{k-1}} \frac{\Psi_{n,k-1}(t) t^l}{z - t^{p+1}} d\sigma_{k-1}(t) = 0, \quad \text{if } d_n - k + 1 + l \not\equiv 0 \pmod{p+1}.$$

and (2.20) follows. □

Definition 2.8. Set $\psi_{n,0} := Q_d$, and for $1 \leq k \leq p$, let $\psi_{n,k}$ be the function analytic in $\mathbb{C} \setminus [a_{k-1}, b_{k-1}]$ defined as

$$\psi_{n,k}(z) = \begin{cases} z \int_{\Gamma_{k-1}} \frac{\Psi_{n,k-1}(t) t^{k-1-\ell}}{z - t^{p+1}} d\sigma_{k-1}(t), & \ell < k, \\ \int_{\Gamma_{k-1}} \frac{\Psi_{n,k-1}(t) t^{p+k-\ell}}{z - t^{p+1}} d\sigma_{k-1}(t), & k \leq \ell. \end{cases}$$

This definition is what one naturally gets by substituting the expressions in (2.21) for s in (2.20), and doing so also yields at once the following corollary.

Corollary 2.9. Suppose $d_n \equiv \ell \pmod{p+1}$ with $0 \leq \ell \leq p$, and define

$$d\sigma_{n,k}(\tau) := \begin{cases} d\sigma_k^*(\tau), & \ell \leq k, \\ \tau d\sigma_k^*(\tau), & k < \ell. \end{cases} \quad (2.22)$$

Then,

$$z^{k-\ell} \Psi_{n,k}(z) = \psi_{n,k}(z^{p+1}), \quad 0 \leq k \leq p, \quad (2.23)$$

and for all $1 \leq k \leq p$,

$$\psi_{n,k}(z) = \begin{cases} z \int_{a_{k-1}}^{b_{k-1}} \frac{\psi_{n,k-1}(\tau)}{z-\tau} d\sigma_{n,k-1}(\tau), & \ell < k, \\ \int_{a_{k-1}}^{b_{k-1}} \frac{\psi_{n,k-1}(\tau)}{z-\tau} d\sigma_{n,k-1}(\tau), & k \leq \ell. \end{cases} \quad (2.24)$$

We have seen that the functions $\Psi_{n,k}$ satisfy orthogonality relations with respect to the hierarchy (2.2). We now show that their associated functions $\psi_{n,k}$ do the same with respect to the hierarchy (2.9).

Proposition 2.10. Let $0 \leq k \leq p-1$ and assume that $d_n \equiv \ell \pmod{p+1}$ with $0 \leq \ell \leq p$. Then the function $\psi_{n,k}$ satisfies the following orthogonality conditions:

$$\int_{a_k}^{b_k} \psi_{n,k}(\tau) \tau^s d\mu_{k,j}(\tau) = 0, \quad \left\lceil \frac{\ell-j}{p+1} \right\rceil \leq s \leq \left\lfloor \frac{n+p\ell-1-j(p+1)}{p(p+1)} \right\rfloor, \quad k \leq j \leq p-1. \quad (2.25)$$

Proof. Let $0 \leq k \leq j \leq p-1$. We start from the orthogonality conditions (2.18), which together with (2.23) and (2.12) gives

$$\begin{aligned} 0 &= \int_{\Gamma_k} \Psi_{n,k}(z) z^l ds_{k,j}(z) = \int_{\Gamma_k} \psi_{n,k}(z^{p+1}) z^{\ell+l-j} z^{j-k} ds_{k,j}(z) \\ &= \int_{a_k}^{b_k} \psi_{n,k}(\tau) \tau^{\frac{\ell+l-j}{p+1}} d\mu_{k,j}(\tau), \end{aligned} \quad (2.26)$$

provided that $0 \leq l \leq \left\lfloor \frac{n-j-1}{p} \right\rfloor$ and $\ell+l-j \equiv 0 \pmod{p+1}$.

If we take l in (2.26) satisfying $\ell+l-j \equiv 0 \pmod{p+1}$, then we can write $l = j - \ell + s(p+1)$ and we obtain the orthogonality conditions

$$\int_{a_k}^{b_k} \psi_{n,k}(\tau) \tau^s d\mu_{k,j}(\tau) = 0, \quad \left\lceil \frac{\ell-j}{p+1} \right\rceil \leq s \leq \left\lfloor \frac{1}{p+1} \left\lfloor \frac{n+p\ell-1-j(p+1)}{p} \right\rfloor \right\rfloor. \quad (2.27)$$

Since for every $x \in \mathbb{R}$,

$$\frac{\lfloor x \rfloor}{p+1} \leq \left\lfloor \frac{\lfloor x \rfloor}{p+1} \right\rfloor + \frac{p}{p+1},$$

it follows that

$$\left\lfloor \frac{\lfloor x \rfloor}{p+1} \right\rfloor \leq \left\lfloor \frac{x}{p+1} \right\rfloor \leq \frac{\lfloor x \rfloor + \{x\}}{p+1} < \frac{\lfloor x \rfloor}{p+1} + \frac{1}{p+1} \leq \left\lfloor \frac{\lfloor x \rfloor}{p+1} \right\rfloor + 1. \quad (2.28)$$

Hence, the first two terms of (2.28) are equal, and the range for s in (2.27) takes the form in (2.25). \square

2.3 Counting the number of orthogonality conditions

Definition 2.11. Let n and ℓ be nonnegative integers with $0 \leq \ell \leq p$. For each $0 \leq j \leq p-1$, let $M_j = M_j(n, \ell)$ be the number of integers s satisfying the inequalities

$$\left\lceil \frac{\ell - j}{p+1} \right\rceil \leq s \leq \left\lfloor \frac{n + p\ell - 1 - j(p+1)}{p(p+1)} \right\rfloor. \quad (2.29)$$

For each $0 \leq k \leq p-1$, we define

$$Z(n, k) = Z(n, \ell, k) := \sum_{j=k}^{p-1} M_j.$$

Also, we convene to set $Z(n, p) := 0$.

Herafter we shall always write $Z(n, k)$ instead of $Z(n, \ell, k)$ because in all future situations the number ℓ will be dependent on n .

It is clear from the definition that for every n and ℓ ,

$$Z(n, k) \geq Z(n, k+1), \quad 0 \leq k \leq p-2,$$

and

$$Z(n, k) - Z(n, k+1) = \# \left\{ s : \left\lceil \frac{\ell - k}{p+1} \right\rceil \leq s \leq \left\lfloor \frac{n + p\ell - 1 - k(p+1)}{p(p+1)} \right\rfloor \right\}. \quad (2.30)$$

Hence, choosing $j = k$ in (2.25), and noticing that

$$\left\lceil \frac{\ell - j}{p+1} \right\rceil = \begin{cases} 0, & \text{if } \ell \leq j, \\ 1, & \text{if } j < \ell, \end{cases} \quad (2.31)$$

we arrive at the following corollary.

Corollary 2.12. Let $0 \leq k \leq p-1$ and assume that $d_n \equiv \ell \pmod{p+1}$ with $0 \leq \ell \leq p$. Then the function $\psi_{n,k}$ satisfies the orthogonality conditions

$$\int_{a_k}^{b_k} \psi_{n,k}(\tau) \tau^s d\sigma_{n,k}(\tau) = 0, \quad 0 \leq s \leq Z(n, k) - Z(n, k+1) - 1. \quad (2.32)$$

Let us fix nonnegative integers n and ℓ , with ℓ satisfying $0 \leq \ell \leq p$. We associate to n and ℓ three numbers α , β , and v , letting α and β be, respectively, the quotient and the remainder in the division of $n + p\ell - 1$ by $p(p+1)$, and letting v be the quotient in the division of β by $p+1$. That is,

$$n + p\ell - 1 = \alpha p(p+1) + \beta, \quad 0 \leq \beta \leq p(p+1) - 1,$$

$$\alpha = \left\lfloor \frac{n + p\ell - 1}{p(p+1)} \right\rfloor, \quad v = \left\lfloor \frac{\beta}{p+1} \right\rfloor. \quad (2.33)$$

Notice that

$$0 \leq v \leq p-1.$$

Lemma 2.13. If $\ell \leq v$, the inequality (2.29) is equivalent to

$$\begin{aligned} 1 \leq s \leq \alpha, & \quad \text{if } 0 \leq j < \ell, \\ 0 \leq s \leq \alpha, & \quad \text{if } \ell \leq j \leq v, \\ 0 \leq s \leq \alpha - 1, & \quad \text{if } v < j \leq p-1, \end{aligned} \quad (2.34)$$

while if $v < \ell$, then (2.29) is equivalent to

$$\begin{aligned} 1 \leq s \leq \alpha, & \quad \text{if } 0 \leq j \leq v, \\ 1 \leq s \leq \alpha - 1, & \quad \text{if } v < j < \ell, \\ 0 \leq s \leq \alpha - 1, & \quad \text{if } \ell \leq j \leq p - 1. \end{aligned} \quad (2.35)$$

Moreover,

$$Z(n, k) = \begin{cases} \left\lceil \frac{n-\ell}{p+1} \right\rceil - k\alpha, & \text{if } k \leq \ell, v, \\ \left\lceil \frac{n-\ell}{p+1} \right\rceil - k\alpha + \ell - v - 1, & \text{if } \ell, v < k, \\ \left\lceil \frac{n-\ell}{p+1} \right\rceil - k(\alpha + 1) + \ell, & \text{if } 0 \leq \ell < k \leq v, \\ \left\lceil \frac{n-\ell}{p+1} \right\rceil - k(\alpha - 1) - v - 1, & \text{if } 0 \leq v < k \leq \ell. \end{cases} \quad (2.36)$$

Proof. We begin by writing (2.29) in the form

$$\left\lceil \frac{\ell - j}{p+1} \right\rceil \leq s \leq \left\lfloor \alpha + \frac{\beta - j(p+1)}{p(p+1)} \right\rfloor. \quad (2.37)$$

Now, since

$$0 \leq v \leq p - 1, \quad v(p+1) \leq \beta < (v+1)(p+1), \quad (2.38)$$

we have

$$-1 < -\frac{(p-1)}{p} \leq \frac{\beta - j(p+1)}{p(p+1)} \leq \frac{p(p+1) - 1}{p(p+1)} < 1,$$

and since α is an integer, this implies that

$$\left\lfloor \alpha + \frac{\beta - j(p+1)}{p(p+1)} \right\rfloor = \begin{cases} \alpha, & \text{if } 0 \leq j \leq v, \\ \alpha - 1, & \text{if } v < j \leq p - 1. \end{cases} \quad (2.39)$$

The inequalities (2.34)-(2.35) follow from (2.31) and (2.39).

As for (2.36), we shall only prove it for the case $k \leq \ell, v$, since the remaining cases listed in (2.36) are proven similarly. If $k \leq \ell \leq v \leq p - 1$, then

$$M_j = \begin{cases} \alpha, & \text{if } k \leq j < \ell, \\ \alpha + 1, & \text{if } \ell \leq j \leq v, \\ \alpha, & \text{if } v < j \leq p - 1, \end{cases}$$

while if $k \leq v < \ell \leq p$, then

$$M_j = \begin{cases} \alpha, & \text{if } k \leq j \leq v, \\ \alpha - 1, & \text{if } v < j < \ell, \\ \alpha, & \text{if } \ell \leq j \leq p - 1. \end{cases}$$

Therefore, if $k \leq \ell, v$, then

$$Z(n, k) = \sum_{j=k}^{p-1} M_j = \alpha(\ell - k) + (\alpha + 1)(v - \ell + 1) + \alpha(p - v - 1) = \alpha p + v - \ell + 1 - k\alpha.$$

Now, using the expression that defines α in (2.33), we find that in either case

$$\alpha p + v - \ell + 1 = \frac{n - \ell + (v+1)(p+1) - (\beta + 1)}{p+1},$$

which together with (2.38) yields

$$\frac{n-\ell}{p+1} \leq \alpha p + v - \ell + 1 \leq \frac{n-\ell}{p+1} + \frac{p}{p+1}.$$

Since $\alpha p + v - \ell + 1$ is an integer, this forces

$$\alpha p + v - \ell + 1 = \left\lceil \frac{n-\ell}{p+1} \right\rceil.$$

□

2.4 AT-system property

The system of continuous functions $u_1(x), \dots, u_n(x)$ is said to be an algebraic Chebyshev system (AT-system) over the interval $[a, b]$ for the set of integers (d_1, \dots, d_n) ($d_j \geq 0$), if for any choice of polynomials $(P_1(x), \dots, P_n(x)) \neq (0, 0, \dots, 0)$, with $\deg(P_j) \leq d_j - 1$, the polynomial combination

$$P_1(x)u_1(x) + \dots + P_n(x)u_n(x)$$

has at most $d_1 + \dots + d_n - 1$ zeros on $[a, b]$. Here and in what follows, a polynomial of degree -1 is understood to be the constant zero function.

Since $\mu_{k,k} = \sigma_k^*$ and $d\mu_{k,j}(t) = t\hat{\mu}_{k+1,j}(t)d\sigma_k^*(t)$ for $k < j \leq p-1$, the orthogonality conditions (2.25) can be equivalently written as $\psi_{n,k}$ being orthogonal to polynomial linear combinations of functions of the form

$$1, t\hat{\mu}_{k+1,k+1}(t), \dots, t\hat{\mu}_{k+1,m}(t), \hat{\mu}_{k+1,m+1}(t), \dots, \hat{\mu}_{k+1,p-1}(t),$$

for some $k \leq m \leq p-1$. We now prove that any such collection of functions forms an AT-system over $[a_k, b_k]$.

Proposition 2.14. *Let k, m be integers such that $0 \leq k \leq m \leq p-1$. For each j in the range $k \leq j \leq p-1$, let P_j be a polynomial of degree at most $d_j - 1$, with $d_j \geq 0$, and suppose that*

$$d_k \geq d_{k+1} \geq \dots \geq d_m \geq d_{m+1} - 1 \geq d_{m+2} - 1 \geq \dots \geq d_{p-1} - 1.$$

If $(P_k, \dots, P_{p-1}) \neq (0, 0, \dots, 0)$, then

$$H(z) = P_k(z) + \sum_{k+1 \leq j \leq m} P_j(z)z\hat{\mu}_{k+1,j}(z) + \sum_{m < j \leq p-1} P_j(z)\hat{\mu}_{k+1,j}(z) \quad (2.40)$$

has at most $D_H := \sum_{j=k}^{p-1} d_j - 1$ zeros in $[a_k, b_k]$.

Proof. The proof is by induction on k . If $k = p-1$, the statement is trivially true, as in this case we simply have $H(z) = P_{p-1}(z)$ and $D_H = d_{p-1} - 1$. Assume that the thesis of the proposition is also true for $k+1$, $0 < k+1 \leq p-1$, but that for the value k , there is a corresponding function H of the form (2.40) with at least $D_H + 1$ zeros in $[a_k, b_k]$.

Then, for this H not all the polynomials P_j corresponding to $k+1 \leq j \leq p-1$, can be simultaneously zero. Let T be a monic polynomial that vanishes at the zeros of H in $[a_k, b_k]$. Then,

$$\frac{H(z)}{T(z)} = O\left(\frac{1}{z^{D_H+2-d_k}}\right), \quad z \rightarrow \infty, \quad (2.41)$$

and H/T is analytic outside the interval $[a_{k+1}, b_{k+1}]$. Let γ be a positively oriented simple contour around the interval $[a_{k+1}, b_{k+1}]$ that leaves outside the zeros of T . From (2.41) we obtain

$$\begin{aligned} 0 &= \frac{1}{2\pi i} \int_{\gamma} \frac{z^u H(z)}{T(z)} dz \\ &= \sum_{k+1 \leq j \leq m} \frac{1}{2\pi i} \int_{\gamma} \frac{z^{u+1} P_j(z)}{T(z)} \left(\int_{a_{k+1}}^{b_{k+1}} \frac{d\mu_{k+1,j}(\tau)}{z - \tau} \right) dz \\ &\quad + \sum_{m < j \leq p-1} \frac{1}{2\pi i} \int_{\gamma} \frac{z^u P_j(z)}{T(z)} \left(\int_{a_{k+1}}^{b_{k+1}} \frac{d\mu_{k+1,j}(\tau)}{z - \tau} \right) dz, \quad u = 0, \dots, D_H - d_k. \end{aligned}$$

Since $\mu_{k+1,k+1} := \sigma_{k+1}^*$, and

$$d\mu_{k+1,j}(\tau) = \left(t \int_{a_{k+2}}^{b_{k+2}} \frac{d\mu_{k+2,j}(s)}{\tau - s} \right) d\sigma_{k+1}^*(\tau), \quad k+1 < j,$$

an application of Cauchy's integral formula and Fubini's theorem yields that if $m = k$, then

$$\int_{a_{k+1}}^{b_{k+1}} \tau^u G(\tau) \frac{d\sigma_{k+1}^*(\tau)}{T(\tau)} = 0, \quad u = 0, \dots, D_H - d_k, \quad (2.42)$$

with

$$G(z) = P_{k+1}(z) + \sum_{k+2 \leq j \leq p-1} P_j(z) z \hat{\mu}_{k+2,j}(z),$$

while if $k+1 \leq m \leq p-1$, then

$$\int_{a_{k+1}}^{b_{k+1}} \tau^u G(\tau) \frac{\tau d\sigma_{k+1}^*(\tau)}{T(\tau)} = 0, \quad u = 0, \dots, D_H - d_k, \quad (2.43)$$

with

$$G(z) = P_{k+1}(z) + \sum_{k+2 \leq j \leq m} z P_j(z) \hat{\mu}_{k+2,j}(z) + \sum_{m < j \leq p-1} P_j(z) \hat{\mu}_{k+2,j}(z).$$

By induction hypothesis, in both cases the function G has at most $D_G = D_H - d_k$ zeros in $[a_{k+1}, b_{k+1}]$, which together with (2.42)-(2.43) implies that G must be identically zero, yielding a contradiction, since at least one of the P_j 's for $k+1 \leq j \leq p-1$ is not identically zero. \square

Corollary 2.15. *Let k, m be integers such that $0 \leq k \leq m \leq p-1$. Let $\{d_j\}_{j=k}^{p-1}$ be a finite sequence of nonnegative integers such that*

$$d_k \geq d_{k+1} \geq \dots \geq d_m \geq d_{m+1} - 1 \geq d_{m+2} - 1 \geq \dots \geq d_{p-1} - 1.$$

Suppose $F \not\equiv 0$ is a function analytic and real-valued on $[a_k, b_k]$, satisfying the orthogonality conditions

$$\int_{a_k}^{b_k} F(\tau) \tau^{s+\delta} d\mu_{k,j}(\tau) = 0, \quad 0 \leq s \leq d_j - 1, \quad k \leq j \leq m, \quad (2.44)$$

$$\int_{a_k}^{b_k} F(\tau) \tau^s d\mu_{k,j}(\tau) = 0, \quad 0 \leq s \leq d_j - 1, \quad m < j \leq p-1, \quad (2.45)$$

where the constant $\delta = 1$ if $m < p-1$ and $d_{m+1} = d_m + 1$, otherwise δ could be taken to be either 1 or 0. Then, F has at least

$$N := \sum_{j=k}^{p-1} d_j$$

zeros of odd multiplicity in (a_k, b_k) .

Proof. Suppose first that $\delta = 0$, so that the sequence $\{d_j\}_{j=k}^{p-1}$ is nonincreasing. In this case the orthogonality conditions (2.44) and (2.45) imply that

$$\int_{a_k}^{b_k} F(\tau) H(\tau) d\sigma_k^*(z) = 0 \quad (2.46)$$

for every H of the form

$$H(z) = P_k(z) + \sum_{j=k+1}^{p-1} P_j(z) z \hat{\mu}_{k+1,j}(z)$$

where P_j is a polynomial of degree at most $d_j - 1$ for each $k \leq j \leq p-1$. Applying Proposition 2.14 (case $m = p-1$), we see that any such function H , not identically zero, has at most $N-1$ zeros in $[a_k, b_k]$. Consequently, if F had a number $D < N$ of zeros with odd multiplicity in (a_k, b_k) , say x_1, \dots, x_D , we could find H with simple zeros at these x_k 's, and with $N-D-1$ zeros (counting multiplicities) at the endpoints of the interval $[a_k, b_k]$. Since H does not admit any more zeros on that closed interval, the integral (2.46) cannot be zero. Therefore, $D \geq N$.

If $\delta = 1$, then (2.44)-(2.45) imply that

$$\int_{a_k}^{b_k} \tau F(\tau) H(\tau) d\sigma_k^*(\tau) = 0$$

for every H of the form

$$H(z) = P_k(z) + \sum_{k+1 \leq j \leq m} P_j(z) z \hat{\mu}_{k+1,j}(z) + \sum_{m < j \leq p-1} P_j(z) \hat{\mu}_{k+1,j}(z).$$

Once again Proposition 2.14 implies that this H has at most $N-1$ zeros in $[a_k, b_k]$, and reasoning as above we conclude that $zF(z)$ (and therefore $F(z)$) has at least N simple zeros in (a_k, b_k) . \square

2.5 Normality of the Nikishin system and zeros of Q_n

The Nikishin system of measures (s_0, \dots, s_{p-1}) is said to be normal provided that the degree of the multi-orthogonal polynomial Q_n is maximal, that is, $d_n = n$ for all $n \geq 0$. We will prove this normality in this section.

Proposition 2.16. *Let n, k , and ℓ be nonnegative integers satisfying that $0 \leq k \leq p-1$, and $d_n \equiv \ell \pmod{p+1}$, $0 \leq \ell \leq p$. Then, the function $\psi_{n,k}$ has at least $Z(n, k)$ zeros with odd multiplicity in the open interval (a_k, b_k) . In particular, it follows that*

$$d_n = n, \quad n \geq 0,$$

that is, the polynomial Q_n has degree n , and the associated polynomial Q_d has exactly

$$d = \frac{n - \ell}{p + 1}$$

zeros, which are all simple and located in (a_0, b_0) .

Proof. According Lemma 2.13, we see that the total number of orthogonality conditions in (2.25) is given by the number $Z(n, k)$, and they can be more specifically written as follows.

If $k \leq \ell \leq v \leq p-1$, then

$$\begin{aligned} \int_{a_k}^{b_k} \psi_{n,k}(\tau) \tau^{s+1} d\mu_{k,j}(\tau) &= 0, & 0 \leq s \leq \alpha-1, & \quad k \leq j < \ell, \\ \int_{a_k}^{b_k} \psi_{n,k}(\tau) \tau^s d\mu_{k,j}(\tau) &= 0, & 0 \leq s \leq \alpha, & \quad \ell \leq j \leq v, \\ \int_{a_k}^{b_k} \psi_{n,k}(\tau) \tau^s d\mu_{k,j}(\tau) &= 0, & 0 \leq s \leq \alpha-1, & \quad v < j \leq p-1. \end{aligned}$$

If $k \leq v < \ell \leq p$, then

$$\begin{aligned} \int_{a_k}^{b_k} \psi_{n,k}(\tau) \tau^{s+1} d\mu_{k,j}(\tau) &= 0, & 0 \leq s \leq \alpha-1, & \quad k \leq j \leq v, \\ \int_{a_k}^{b_k} \psi_{n,k}(\tau) \tau^{s+1} d\mu_{k,j}(\tau) &= 0, & 0 \leq s \leq \alpha-2, & \quad v < j < \ell, \\ \int_{a_k}^{b_k} \psi_{n,k}(\tau) \tau^s d\mu_{k,j}(\tau) &= 0, & 0 \leq s \leq \alpha-1, & \quad \ell \leq j \leq p-1. \end{aligned}$$

If $0 \leq \ell, v < k$, then

$$\int_{a_k}^{b_k} \psi_{n,k}(\tau) \tau^s d\mu_{k,j}(\tau) = 0, \quad 0 \leq s \leq \alpha-1, \quad k \leq j \leq p-1.$$

If $0 \leq \ell < k \leq v$, then

$$\begin{aligned} \int_{a_k}^{b_k} \psi_{n,k}(\tau) \tau^s d\mu_{k,j}(\tau) &= 0, & 0 \leq s \leq \alpha, & \quad k \leq j \leq v, \\ \int_{a_k}^{b_k} \psi_{n,k}(\tau) \tau^s d\mu_{k,j}(\tau) &= 0, & 0 \leq s \leq \alpha-1, & \quad v < j \leq p-1, \end{aligned}$$

and finally, if $v < k \leq \ell$, then

$$\begin{aligned} \int_{a_k}^{b_k} \psi_{n,k}(\tau) \tau^{s+1} d\mu_{k,j}(\tau) &= 0, & 0 \leq s \leq \alpha-2, & \quad k \leq j < \ell, \\ \int_{a_k}^{b_k} \psi_{n,k}(\tau) \tau^s d\mu_{k,j}(\tau) &= 0, & 0 \leq s \leq \alpha-1, & \quad \ell \leq j \leq p-1. \end{aligned}$$

By Corollary 2.15, in each case we deduce that $\psi_{n,k}$ has at least $Z(n, k)$ zeros of odd multiplicity in (a_k, b_k) .

Particularly, for $k = 0$, $\psi_{n,0} = \mathcal{Q}_d$ has at least $Z(n, 0) = \left\lceil \frac{n-\ell}{p+1} \right\rceil$ zeros of odd multiplicity in (a_0, b_0) . Hence,

$$\left\lceil \frac{n-\ell}{p+1} \right\rceil \leq d = \frac{d_n - \ell}{p+1} \leq \frac{n-\ell}{p+1},$$

finishing the proof of the proposition. \square

In Figures 1 and 2 we have plotted the zeros of Q_{29} , Q_{30} , and Q_{45} corresponding to the Nikishin system $\mathcal{N}(\sigma_0, \sigma_1)$ generated by the $p = 2$ measures

$$\begin{aligned} d\sigma_0(t) &= |t|^2 |dt|, \quad t \in \Gamma_0 = [0, 1] \cup e^{2\pi i/3}[0, 1] \cup e^{4\pi i/3}[0, 1], \\ d\sigma_1(t) &= |t|^2 |dt|, \quad t \in \Gamma_1 = [-2, -1] \cup e^{2\pi i/3}[-2, -1] \cup e^{4\pi i/3}[-2, -1]. \end{aligned}$$

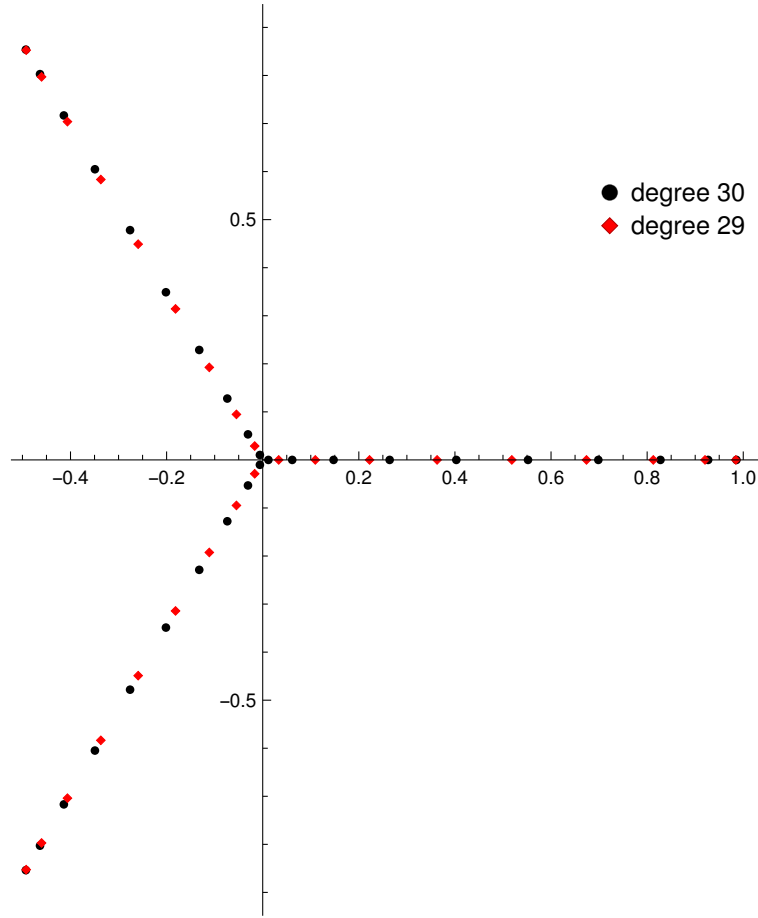


Figure 1: Zeros of Q_{29} and Q_{30} for $p = 2$, where $d\sigma_0(t) = |t|^2|dt|$, $t \in \Gamma_0$, $[a_0, b_0] = [0, 1]$, and $d\sigma_1(t) = |t|^2|dt|$, $t \in \Gamma_1$, $[a_1, b_1] = [-2, -1]$. The zero at the origin of Q_{29} is omitted.

2.6 Alternative formulas for the quantities $Z(n, k)$

Knowing that the degree of Q_n is n , we seek to express the quantity $Z(n, k) = Z(n, \ell, k)$ with $n \equiv \ell \pmod{p+1}$ in terms of the remainder of n when divided by p .

Proposition 2.17. *Let n be any nonnegative integer. Suppose $n \equiv \ell \pmod{p+1}$ and $n \equiv r \pmod{p}$, $0 \leq \ell \leq p$, $0 \leq r \leq p-1$, and let*

$$\lambda = \left\lfloor \frac{n}{p(p+1)} \right\rfloor.$$

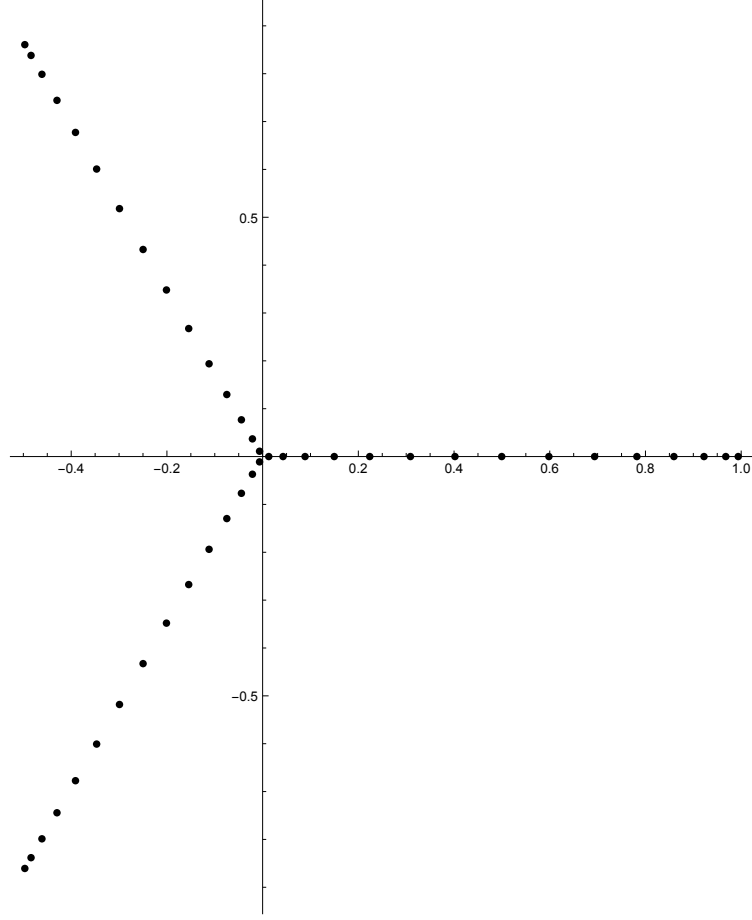


Figure 2: Zeros of Q_{45} for $p = 2$, where $d\sigma_0(t) = |t|^2|dt|$, $t \in \Gamma_0$, $[a_0, b_0] = [0, 1]$, and $d\sigma_1(t) = |t|^2|dt|$, $t \in \Gamma_1$, $[a_1, b_1] = [-2, -1]$.

Then, for every $k = 0, \dots, p-1$ we have

$$Z(n, k) = \begin{cases} \left\lfloor \frac{n}{p+1} \right\rfloor - k\lambda, & k < \ell \leq r, \\ \left\lfloor \frac{n}{p+1} \right\rfloor - k(\lambda + 1) + \ell, & \ell \leq k < r, \\ \left\lfloor \frac{n}{p+1} \right\rfloor - k\lambda + \ell - r, & \ell \leq r \leq k, \\ \left\lfloor \frac{n}{p+1} \right\rfloor - k(\lambda + 1), & k < r < \ell, \\ \left\lfloor \frac{n}{p+1} \right\rfloor - k\lambda - r, & r \leq k < \ell, \\ \left\lfloor \frac{n}{p+1} \right\rfloor - k(\lambda + 1) + \ell - r, & r < \ell \leq k. \end{cases} \quad (2.47)$$

The first three cases of (2.47) correspond to $\ell \leq r$, while the last three correspond to $r < \ell$. In particular,

$$Z(n, k) = \frac{n(p-k)}{p(p+1)} + O(1), \quad n \rightarrow \infty. \quad (2.48)$$

Proof. Let us write $n = mp + r$ with $m \geq 0$, and

$$\begin{aligned} m &= \lambda(p+1) + q, \quad 0 \leq q \leq p, \\ n &= \lambda p(p+1) + pq + r, \quad 0 \leq pq + r \leq p(p+1) - 1, \end{aligned} \quad (2.49)$$

so that

$$\lambda = \left\lfloor \frac{n}{p(p+1)} \right\rfloor.$$

From the equalities

$$n = (\lambda p + q)(p+1) + r - q = (\lambda p + q - 1)(p+1) + p + 1 + r - q,$$

we see that the remainder ℓ in the division of n by $p+1$ is given by

$$\ell = \begin{cases} r - q, & \ell \leq r, \\ p + 1 + r - q, & r < \ell. \end{cases} \quad (2.50)$$

Combining (2.30), (2.49), and (2.50), we get that for all $0 \leq j \leq p-1$,

$$\begin{aligned} Z(n, j) - Z(n, j+1) &= \lambda + \left\lfloor \frac{(r-j)(p+1)-1}{p(p+1)} \right\rfloor + \begin{cases} 0, & \ell \leq r, \\ 1, & r < \ell \end{cases} \\ &\quad + \begin{cases} 1, & \ell \leq j, \\ 0, & j < \ell. \end{cases} \end{aligned}$$

Hence,

$$Z(n, j) - Z(n, j+1) = \begin{cases} \lambda, & j < \ell \leq r, \\ \lambda + 1, & \ell \leq j < r, \\ \lambda, & \ell \leq r \leq j, \\ \lambda + 1, & j < r < \ell, \\ \lambda, & r \leq j < \ell, \\ \lambda + 1, & r < \ell \leq j. \end{cases} \quad (2.51)$$

The first three cases of (2.51) correspond to $\ell \leq r$, while the last three correspond to $r < \ell$.

Now, $Z(n, k) = \sum_{j=k}^{p-1} Z(n, j) - Z(n, j+1)$, and so we get that

$$Z(n, k) = \begin{cases} \lambda(\ell - k) + (r - \ell)(\lambda + 1) + (p - r)\lambda, & k < \ell \leq r, \\ (\lambda + 1)(r - k) + \lambda(p - r), & \ell \leq k < r, \\ \lambda(p - k), & \ell \leq r \leq k, \\ (\lambda + 1)(r - k) + \lambda(\ell - r) + (\lambda + 1)(p - \ell), & k < r < \ell, \\ \lambda(\ell - k) + (\lambda + 1)(p - \ell), & r \leq k < \ell, \\ (\lambda + 1)(p - k), & r < \ell \leq k. \end{cases}$$

Simplifying these expressions, we obtain (2.47) if we take into account that

$$\left\lfloor \frac{n}{p+1} \right\rfloor = \frac{n - \ell}{p+1} = \begin{cases} p\lambda + q, & \ell \leq r, \\ p\lambda + q - 1, & r < \ell. \end{cases}$$

Finally, from (2.47) we deduce that

$$Z(n, k) = \left\lfloor \frac{n}{p+1} \right\rfloor - k \left\lfloor \frac{n}{p(p+1)} \right\rfloor + O(1) = \frac{n}{p+1} - k \frac{n}{p(p+1)} + O(1),$$

which gives (2.48). \square

2.7 Order of decay and zeros of the functions of the second kind

Proposition 2.18. *Let $1 \leq k \leq p$, and suppose that $n \equiv \ell \pmod{p+1}$. Then, as $z \rightarrow \infty$,*

$$\psi_{n,k}(z) = O(z^{-N(n,k)}), \quad (2.52)$$

where

$$N(n,k) = \begin{cases} Z(n,k-1) - Z(n,k), & \ell < k, \\ Z(n,k-1) - Z(n,k) + 1, & k \leq \ell, \end{cases}$$

recall $Z(n,p) = 0$.

Proof. From (2.24) we see that for $1 \leq k \leq p$, the Laurent expansion of $\psi_{n,k}$ at infinity has the following form. If $\ell < k$,

$$\psi_{n,k}(z) = \sum_{s=0}^{\infty} z^{-s} \int_{a_{k-1}}^{b_{k-1}} \psi_{n,k-1}(\tau) \tau^s d\sigma_{k-1}^*(\tau), \quad (2.53)$$

while if $k \leq \ell$,

$$\psi_{n,k}(z) = \sum_{s=1}^{\infty} z^{-s} \int_{a_{k-1}}^{b_{k-1}} \psi_{n,k-1}(\tau) \tau^s d\sigma_{k-1}^*(\tau). \quad (2.54)$$

Now, (2.32) states that that if $\ell < k$,

$$\int_{a_{k-1}}^{b_{k-1}} \psi_{n,k-1}(\tau) \tau^s d\sigma_{k-1}^*(\tau) = 0, \quad 0 \leq s \leq Z(n,k-1) - Z(n,k) - 1,$$

while if $k \leq \ell$,

$$\int_{a_{k-1}}^{b_{k-1}} \psi_{n,k-1}(\tau) \tau^s d\sigma_{k-1}^*(\tau) = 0, \quad 1 \leq s \leq Z(n,k-1) - Z(n,k),$$

which combined with (2.53)-(2.54) yields (2.52). \square

We are now in position to prove the following result.

Proposition 2.19. *For each $n \geq 0$ and $k = 0, \dots, p-1$, the function $\psi_{n,k}$ has exactly $Z(n,k)$ zeros in $\mathbb{C} \setminus ([a_{k-1}, b_{k-1}] \cup \{0\})$; they are all simple and lie in the open interval (a_k, b_k) . The function $\psi_{n,p}$ has no zeros in $\mathbb{C} \setminus ([a_{p-1}, b_{p-1}] \cup \{0\})$.*

Proof. The proof is by induction on k . It was already shown in Proposition 2.16 that the polynomial $\psi_{n,0} = Q_d$ has degree $Z(n,0) = \frac{n-\ell}{p+1}$, all its zeros are simple and lie in the interval (a_0, b_0) .

Let us assume that the result holds for $k-1$, $k \geq 1$, but that $\psi_{n,k}$ has at least $Z(n,k)+1$ zeros in $\mathbb{C} \setminus ([a_{k-1}, b_{k-1}] \cup \{0\})$, counting multiplicities. Let $P_{n,k}(z)$ denote the monic polynomial whose zeros are the zeros of $\psi_{n,k}$ in $\mathbb{C} \setminus ([a_{k-1}, b_{k-1}] \cup \{0\})$. Since $\psi_{n,k}(\bar{z}) = \overline{\psi_{n,k}(z)}$, the complex zeros of $\psi_{n,k}$, if any, must come in conjugate pairs, so $P_{n,k}$ is a polynomial with real coefficients with $\deg(P_{n,k}) \geq Z(n,k) + 1$.

We first assume that $\ell < k$. It follows from (2.52) that

$$\frac{\psi_{n,k}(z)}{z P_{n,k}(z)} = O\left(\frac{1}{z^{Z(n,k-1)+2}}\right), \quad z \rightarrow \infty, \quad (2.55)$$

and this function is analytic outside $[a_{k-1}, b_{k-1}]$. Let γ be a closed Jordan curve that surrounds $[a_{k-1}, b_{k-1}]$ and leaves the zeros of $P_{n,k}$ outside. Then, it follows from (2.55) and (2.24) that for $j = 0, \dots, Z(n, k-1)$, we have

$$\begin{aligned} 0 &= \frac{1}{2\pi i} \int_{\gamma} \frac{z^j \psi_{n,k}(z)}{z P_{n,k}(z)} dz = \frac{1}{2\pi i} \int_{\gamma} \frac{z^j}{P_{n,k}(z)} \int_{a_{k-1}}^{b_{k-1}} \frac{\psi_{n,k-1}(\tau)}{z - \tau} d\sigma_{k-1}^*(\tau) dz \\ &= \int_{a_{k-1}}^{b_{k-1}} \psi_{n,k-1}(\tau) \tau^j \frac{d\sigma_{k-1}^*(\tau)}{P_{n,k}(\tau)}, \end{aligned}$$

where we applied Cauchy's theorem, Cauchy's integral formula and Fubini's theorem. The above orthogonality conditions of $\psi_{n,k-1}$ with respect to the measure $\frac{d\sigma_{k-1}^*(\tau)}{P_{n,k}(\tau)}$ imply that $\psi_{n,k-1}$ has at least $Z(n, k-1) + 1$ zeros in (a_{k-1}, b_{k-1}) , contrary to our initial hypothesis.

Let us assume now that $k \leq \ell$. Then this time

$$\frac{\psi_{n,k}(z)}{P_{n,k}(z)} = O\left(\frac{1}{z^{Z(n,k-1)+2}}\right), \quad z \rightarrow \infty,$$

and this function is again analytic outside $[a_{k-1}, b_{k-1}]$. The same argument above leads now to the orthogonality conditions

$$\int_{a_{k-1}}^{b_{k-1}} \psi_{n,k-1}(\tau) \tau^j \frac{\tau d\sigma_{k-1}^*(\tau)}{P_{n,k}(\tau)} = 0, \quad j = 0, \dots, Z(n, k-1),$$

forcing $\psi_{n,k-1}$ to have at least $Z(n, k-1) + 1$ zeros in (a_{k-1}, b_{k-1}) , again a contradiction.

In conclusion, the function $\psi_{n,k}$ has at most $Z(n, k)$ zeros in $\mathbb{C} \setminus ([a_{k-1}, b_{k-1}] \cup \{0\})$. This together with Proposition 2.16 gives the result. \square

For the asymptotic analysis that will be performed later it is crucial to consider the polynomials whose zeros coincide with those of the functions $\psi_{n,k}$. We introduce now a notation for these polynomials.

Definition 2.20. For any integers $n \geq 0$ and k with $0 \leq k \leq p-1$, let $P_{n,k}$ denote the monic polynomial whose zeros are the zeros of $\psi_{n,k}$ in (a_k, b_k) . For convenience we also define the polynomials $P_{n,-1} \equiv 1$, $P_{n,p} \equiv 1$.

Hence by Proposition 2.19 we know that $P_{n,k}$ has degree $Z(n, k)$ and all its zeros are simple. Note that $P_{n,0} = \psi_{n,0}$.

Proposition 2.21. Let $0 \leq k \leq p-1$ and $n \equiv \ell \pmod{p+1}$, $0 \leq \ell \leq p$. Then, the function $\psi_{n,k}$ satisfies the following orthogonality conditions:

$$\int_{a_k}^{b_k} \psi_{n,k}(\tau) \tau^s \frac{d\sigma_{n,k}(\tau)}{P_{n,k+1}(\tau)} = 0, \quad s = 0, \dots, Z(n, k) - 1. \quad (2.56)$$

Proof. For $0 \leq k \leq p-2$, these orthogonality conditions follow immediately from the definition of the polynomials $P_{n,k}$ and the argument given in the proof of Proposition 2.19. For $k = p-1$, (2.56) follows from (2.25) and (2.36), since $\left\lceil \frac{\ell-(p-1)}{p+1} \right\rceil = 0$ if $\ell \leq p-1$ and $\left\lceil \frac{\ell-(p-1)}{p+1} \right\rceil = 1$ if $\ell = p$. \square

Corollary 2.22. Let $0 \leq k \leq p-1$, and let I be any connected component of $[a_k, b_k] \setminus \text{supp}(\sigma_k^*)$. Then the polynomial $P_{n,k}$ has at most one zero in the closure of I .

Proof. Suppose that $P_{n,k}$ has two distinct zeros τ_1 and τ_2 in \bar{I} and assume that $\ell \leq k$ (the case $k < \ell$ follows along the same lines). Then according to (2.56) we have

$$\int_{a_k}^{b_k} \psi_{n,k}(\tau) \frac{P_{n,k}(\tau)}{(\tau - \tau_1)(\tau - \tau_2)} \frac{d\sigma_k^*(\tau)}{P_{n,k+1}(\tau)} = 0, \quad (2.57)$$

since $\frac{P_{n,k}(\tau)}{(\tau-\tau_1)(\tau-\tau_2)}$ is a polynomial of degree $Z(n,k) - 2$. On the other hand, the function

$$\frac{\psi_{n,k}(\tau) P_{n,k}(\tau)}{(\tau - \tau_1)(\tau - \tau_2)}$$

has constant sign and finitely many zeros on $\text{supp}(\sigma_k^*)$, therefore its integral with respect to the measure $d\sigma_k^*(\tau)/P_{n,k+1}(\tau)$ should be different from zero. This contradicts (2.57). \square

2.8 The auxiliary functions $H_{n,k}$

We now introduce certain functions that will play an important role in the analysis that will follow.

Definition 2.23. For integers $n \geq 0$ and $0 \leq k \leq p$, set

$$H_{n,k}(z) := \frac{P_{n,k-1}(z) \psi_{n,k}(z)}{P_{n,k}(z)}. \quad (2.58)$$

Note that $H_{n,0} \equiv 1$. Since the zeros of $P_{n,k}$ are zeros of $\psi_{n,k}$ outside $[a_{k-1}, b_{k-1}]$, we have

$$H_{n,k} \in \mathcal{H}(\mathbb{C} \setminus [a_{k-1}, b_{k-1}]), \quad 1 \leq k \leq p.$$

Putting together (2.22), (2.56), and (2.58), we readily obtain the following result.

Proposition 2.24. For any $k = 0, \dots, p-1$, the polynomial $P_{n,k}$ satisfies the following orthogonality conditions:

$$\int_{a_k}^{b_k} P_{n,k}(\tau) \tau^s \frac{H_{n,k}(\tau) d\sigma_{n,k}(\tau)}{P_{n,k-1}(\tau) P_{n,k+1}(\tau)} = 0, \quad s = 0, \dots, Z(n,k) - 1. \quad (2.59)$$

Recall that $P_{n,-1}, P_{n,p} \equiv 1$.

2.9 Integral representation of the functions $H_{n,k}$

We prove now a formula analogous to (2.24) for the functions $H_{n,k}$.

Proposition 2.25. Let $1 \leq k \leq p$ and $n \equiv \ell \pmod{p+1}$, $0 \leq \ell \leq p$. Then,

$$H_{n,k}(z) = \begin{cases} z \int_{a_{k-1}}^{b_{k-1}} \frac{P_{n,k-1}^2(\tau)}{z-\tau} \frac{H_{n,k-1}(\tau) d\sigma_{n,k-1}(\tau)}{P_{n,k-2}(\tau) P_{n,k}(\tau)}, & \ell < k, \\ \int_{a_{k-1}}^{b_{k-1}} \frac{P_{n,k-1}^2(\tau)}{z-\tau} \frac{H_{n,k-1}(\tau) d\sigma_{n,k-1}(\tau)}{P_{n,k-2}(\tau) P_{n,k}(\tau)}, & k \leq \ell. \end{cases} \quad (2.60)$$

Proof. We know by (2.59) that for any polynomial Q with $\deg(Q) \leq Z(n,k-1)$, $1 \leq k \leq p$, we have

$$\int_{a_{k-1}}^{b_{k-1}} \frac{Q(z) - Q(\tau)}{z - \tau} P_{n,k-1}(\tau) \frac{H_{n,k-1}(\tau) d\sigma_{n,k-1}(\tau)}{P_{n,k-2}(\tau) P_{n,k}(\tau)} = 0. \quad (2.61)$$

If we take in particular $Q = P_{n,k-1}$ in (2.61), then we obtain

$$P_{n,k-1}(z) \int_{a_{k-1}}^{b_{k-1}} \frac{P_{n,k-1}(\tau)}{z - \tau} \frac{H_{n,k-1}(\tau) d\sigma_{n,k-1}(\tau)}{P_{n,k-2}(\tau) P_{n,k}(\tau)} = \int_{a_{k-1}}^{b_{k-1}} \frac{P_{n,k-1}^2(\tau)}{z - \tau} \frac{H_{n,k-1}(\tau) d\sigma_{n,k-1}(\tau)}{P_{n,k-2}(\tau) P_{n,k}(\tau)}.$$

Since $Z(n,k) \leq Z(n,k-1)$, we can apply (2.61) for $Q = P_{n,k}$ and we get

$$\int_{a_{k-1}}^{b_{k-1}} \frac{P_{n,k-1}(\tau)}{z - \tau} \frac{H_{n,k-1}(\tau) d\sigma_{n,k-1}(\tau)}{P_{n,k-2}(\tau) P_{n,k}(\tau)} = \frac{1}{P_{n,k}(z)} \int_{a_{k-1}}^{b_{k-1}} \frac{\psi_{n,k-1}(\tau)}{z - \tau} d\sigma_{n,k-1}(\tau).$$

From the last two identities we deduce that for $k = 1, \dots, p$,

$$\frac{1}{P_{n,k}(z)} \int_{a_{k-1}}^{b_{k-1}} \frac{\psi_{n,k-1}(\tau)}{z - \tau} d\sigma_{n,k-1}(\tau) = \frac{1}{P_{n,k-1}(z)} \int_{a_{k-1}}^{b_{k-1}} \frac{P_{n,k-1}^2(\tau)}{z - \tau} \frac{H_{n,k-1}(\tau) d\sigma_{n,k-1}(\tau)}{P_{n,k-2}(\tau) P_{n,k}(\tau)}. \quad (2.62)$$

In virtue of (2.24), we have

$$\int_{a_{k-1}}^{b_{k-1}} \frac{\psi_{n,k-1}(\tau)}{z - \tau} d\sigma_{n,k-1}(\tau) = \begin{cases} z^{-1} \psi_{n,k}(z), & \ell < k, \\ \psi_{n,k}(z), & \text{if } k \leq \ell. \end{cases} \quad (2.63)$$

Hence the result follows from (2.62), (2.63) and (2.58). \square

In what follows, we shall use the notation $\text{sign}(f, I)$ to mean the sign of the function f on the interval I , and Δ_k shall denote the open interval (a_k, b_k) .

Corollary 2.26. *Let $1 \leq k \leq p-1$ and $n \equiv \ell \pmod{p+1}$, $0 \leq \ell \leq p$. Then, with the convention that $Z(n, -1) = 0$, we have*

$$\text{sign}(H_{n,k}, \Delta_k) = \begin{cases} (-1)^{(k+1)[Z(n,k-2)-Z(n,k)]} \text{sign}(H_{n,k-1}, \Delta_{k-1}), & \ell < k, \\ (-1)^{1+(k+1)[Z(n,k-2)-Z(n,k)]} \text{sign}(H_{n,k-1}, \Delta_{k-1}), & k \leq \ell. \end{cases} \quad (2.64)$$

Proof. Suppose first that $k \leq \ell$. Then, by (2.60) and (2.22),

$$H_{n,k}(z) = \int_{\Delta_{k-1}} \frac{P_{n,k-1}^2(\tau)}{z - \tau} \frac{H_{n,k-1}(\tau) \tau d\sigma_{k-1}^*(\tau)}{P_{n,k-2}(\tau) P_{n,k}(\tau)}.$$

If k is even, Δ_{k-1} lies in $(-\infty, 0)$, while Δ_{k-2} and Δ_k lie in $(0, \infty)$. Since the monic polynomials $P_{n,k-2}$, $P_{n,k}$ have their zeros in Δ_{k-2} , Δ_k , respectively, and $\deg(P_{n,k}) = Z(n, k)$, the above equality gives

$$\text{sign}(H_{n,k}, \Delta_k) = (-1)^{Z(n,k-2)-Z(n,k)+1} \text{sign}(H_{n,k-1}, \Delta_{k-1}).$$

If k is odd, Δ_{k-1} lies in $(0, \infty)$, $P_{n,k-2}$ and $P_{n,k}$ are both positive in Δ_{k-1} , so that

$$\text{sign}(H_{n,k}, \Delta_k) = -\text{sign}(H_{n,k-1}, \Delta_{k-1}).$$

Suppose now that $\ell < k$. Then, by (2.60) and (2.22),

$$H_{n,k}(z) = z \int_{\Delta_{k-1}} \frac{P_{n,k-1}^2(\tau)}{z - \tau} \frac{H_{n,k-1}(\tau) d\sigma_{k-1}^*(\tau)}{P_{n,k-2}(\tau) P_{n,k}(\tau)},$$

so that if k is even,

$$\text{sign}(H_{n,k}, \Delta_k) = (-1)^{Z(n,k-2)-Z(n,k)} \text{sign}(H_{n,k-1}, \Delta_{k-1}),$$

while for k odd,

$$\text{sign}(H_{n,k}, \Delta_k) = \text{sign}(H_{n,k-1}, \Delta_{k-1}).$$

\square

3 Recurrence relation and positivity of the recurrence coefficients

Proposition 3.1. *The polynomials Q_n satisfy the following three-term recurrence relation of order $p+1$:*

$$zQ_n(z) = Q_{n+1}(z) + a_n Q_{n-p}(z), \quad n \geq p, \quad a_n \in \mathbb{R}, \quad (3.1)$$

where

$$Q_\ell(z) = z^\ell, \quad \ell = 0, \dots, p. \quad (3.2)$$

Proof. The equation (3.2) is clear since we know that if $n = d(p+1) + \ell$, $0 \leq \ell \leq p$, then $Q_n(z) = z^\ell Q_d(z^{p+1})$ for some monic polynomial Q_d of degree d . Moreover, this also implies that for $n \geq p$, $zQ_n(z) - Q_{n+1}(z) = c_{n-p}z^{n-p} + \dots$. Thus, we can write

$$zQ_n = Q_{n+1} + \sum_{j=0}^{n-p} b_j Q_j, \quad (3.3)$$

for some real coefficients $\{b_j\}_{j=0}^{n-p}$. The goal is to show that

$$b_0 = b_1 = \dots = b_{n-p-1} = 0. \quad (3.4)$$

Assume that $n = mp + k$, $0 \leq k \leq p-1$. If we integrate (3.3) term by term with respect to the first measure s_0 of the Nikishin system, we observe that the only non-vanishing integral is $\int Q_0 ds_0$, and consequently $b_0 = 0$. Integrating (3.3) successively with respect to s_j we obtain $b_j = 0$ for $j = 1, \dots, p-1$.

In general, one proves inductively that for all l such that $0 \leq l \leq m-2$, we have

$$b_{lp} = b_{lp+1} = \dots = b_{lp+(p-1)} = 0. \quad (3.5)$$

The case $l = 0$ was described above. Assume now that all coefficients b_s in (3.3) are zero for $s < lp$. If we multiply (3.3) by z^l and integrate with respect to s_0 , then the only non-vanishing integral in the resulting expression is $\int z^l Q_{lp}(z) ds_0(z)$. Indeed, all other integrals vanish because of the orthogonality conditions, and $\int z^l Q_{lp}(z) ds_0(z) = 0$ would imply that Q_{lp+1} and Q_{lp} satisfy the same orthogonality conditions, violating the normality of the Nikishin system. So $b_{lp} = 0$. Integrating successively with respect to the rest of the measures s_j one obtains (3.5). The remaining part of (3.4) is

$$b_{(m-1)p} = b_{(m-1)p+1} = \dots = b_{(m-1)p+k-1} = 0,$$

which is proved multiplying by z^{m-1} and integrating with respect to s_0, \dots, s_{k-1} . \square

We now show that the functions of the second kind satisfy a similar recurrence relation.

Proposition 3.2. *Let a_n , $n \geq p$, be the coefficients of the recurrence relation (3.1). For every $n \geq p$, $0 \leq k \leq p$, we have*

$$z\Psi_{n,k}(z) = \Psi_{n+1,k}(z) + a_n \Psi_{n-p,k}(z), \quad (3.6)$$

and if $n \equiv \ell \pmod{p+1}$, $0 \leq \ell \leq p-1$, then

$$\psi_{n,k}(z) = \psi_{n+1,k}(z) + a_n \psi_{n-p,k}(z), \quad (3.7)$$

while if $n \equiv p \pmod{p+1}$, then

$$z\psi_{n,k}(z) = \psi_{n+1,k}(z) + a_n \psi_{n-p,k}(z). \quad (3.8)$$

Proof. For $k = 0$, by definition, $\Psi_{n,0} = Q_n$, and so (3.6) reduces to (3.1). Let us then assume that (3.6) holds for some $0 \leq k \leq p-1$. Then, by the very definition of $\Psi_{n,k}$, we have

$$\begin{aligned} \int_{\Gamma_k} \frac{t\Psi_{n,k}(t)d\sigma_k(t)}{z-t} &= \int_{\Gamma_k} \frac{\Psi_{n+1,k}(t)d\sigma_k(t)}{z-t} + a_n \int_{\Gamma_k} \frac{\Psi_{n-p,k}(t)d\sigma_k(t)}{z-t} \\ &= \Psi_{n+1,k+1}(z) + a_n \Psi_{n-p,k+1}(z). \end{aligned}$$

Now, from Proposition 2.5, we know that

$$\int_{\Gamma_k} \Psi_{n,k}(z)z^l d\sigma_k(z) = 0, \quad 0 \leq l \leq \left\lfloor \frac{n-k-1}{p} \right\rfloor,$$

and since $n-k-1 \geq p-1-k \geq 0$, we have that $\int_{\Gamma_k} \Psi_{n,k}(z)d\sigma_k(z) = 0$ and so

$$\int_{\Gamma_k} \frac{t\Psi_{n,k}(t)d\sigma_k(t)}{z-t} = z \int_{\Gamma_k} \frac{\Psi_{n,k}(t)d\sigma_k(t)}{z-t} - \int_{\Gamma_k} \Psi_{n,k}(t)d\sigma_k(t) = z\Psi_{n,k+1}(z).$$

Now, using (2.23) in (3.6) we find that for $n \geq p$, $0 \leq k \leq p$,

$$z^{\ell_{n+1}-k}\psi_{n,k}(z^{p+1}) = z^{\ell_{n+1}-k}\psi_{n+1,k}(z^{p+1}) + a_n z^{\ell_{n-p}-k}\psi_{n-p,k}(z^{p+1}).$$

where we are using the notation ℓ_n to mean the remainder of n in the division by $p+1$, $0 \leq \ell_n \leq p$.

The relations (3.7)-(3.8) then follow from the fact that $\ell_{n+1} = \ell_{n-p} = \ell_n + 1$ when $\ell_n \leq p-1$, while $\ell_{n+1} = \ell_{n-p} = 0$ when $\ell_n = p$. \square

Lemma 3.3. *Let $n \geq p$ and suppose that $n \equiv k \pmod{p}$, $0 \leq k \leq p-1$, and $n \equiv \ell \pmod{p+1}$, $0 \leq \ell \leq p$. Then*

$$\int_{a_k}^{b_k} \psi_{n+1,k}(\tau) \tau^{Z(n,k)-Z(n,k+1)} d\sigma_{n,k}(\tau) = 0, \quad \ell \leq p-1, \quad (3.9)$$

$$\int_{a_k}^{b_k} \psi_{n+1,k}(\tau) \tau^{Z(n,k)-Z(n,k+1)-1} d\sigma_{n,k}(\tau) = 0, \quad \ell = p, \quad (3.10)$$

and

$$\tau^{Z(n-p,k)-Z(n-p,k+1)} d\sigma_{n-p,k}(\tau) = \begin{cases} \tau^{Z(n,k)-Z(n,k+1)} d\sigma_{n,k}(\tau), & \ell \leq p-1, \\ \tau^{Z(n,k)-Z(n,k+1)-1} d\sigma_{n,k}(\tau), & \ell = p. \end{cases} \quad (3.11)$$

Proof. Obviously,

$$n+1 \equiv \begin{cases} \ell+1 \pmod{p+1}, & \ell \leq p-1, \\ 0 \pmod{p+1}, & \ell = p. \end{cases} \quad (3.12)$$

With this in mind, we readily get from (2.22) that (3.9)-(3.10) are equivalent to

$$\int_{a_k}^{b_k} \psi_{n+1,k}(\tau) \tau^{Z(n,k)-Z(n,k+1)} d\sigma_{n+1,k}(\tau) = 0, \quad \ell \neq k, \quad (3.13)$$

$$\int_{a_k}^{b_k} \psi_{n+1,k}(\tau) \tau^{Z(n,k)-Z(n,k+1)-1} d\sigma_{n+1,k}(\tau) = 0, \quad \ell = k. \quad (3.14)$$

Suppose $n \geq p$, $n \equiv k \pmod{p}$, $0 \leq k \leq p-1$. Then,

$$n+1 \equiv \begin{cases} k+1 \pmod{p}, & k < p-1, \\ 0 \pmod{p}, & k = p-1. \end{cases} \quad (3.15)$$

We then use (2.51) to analyze all possible cases emanating from (3.12) and (3.15). Using the notation

$$\lambda(n) = \left\lfloor \frac{n}{p(p+1)} \right\rfloor,$$

we find that

$$Z(n, k) - Z(n, k+1) = \lambda(n),$$

$$Z(n+1, k) - Z(n+1, k+1) = \begin{cases} \lambda(n+1), & \ell = k, \\ \lambda(n+1), & \ell = p, k = p-1, \\ \lambda(n+1) + 1, & \text{otherwise.} \end{cases}$$

Now, $\lambda(n+1) = \lambda(n)$ if $n+1$ is not a multiple of $p(p+1)$, and $\lambda(n+1) = \lambda(n) + 1$ otherwise. The latter case holds exactly when $\ell = p$ and $k = p-1$. We then conclude that in all instances the exponent of τ in (3.13)-(3.14) equals $Z(n+1, k) - Z(n+1, k+1) - 1$. Notice that this quantity is non-negative since the smallest integer $n \geq p$ satisfying that $\ell = k$ (i.e., having the same remainder when divided by p and by $p+1$) is $n = p(p+1)$. This together with (2.32) yields (3.9)-(3.10).

Now, both n and $n-p$ leave the same remainder k when they are divided by p . If $\ell \leq p-1$, then $n-p \equiv \ell+1 \pmod{p+1}$, and (2.51) yields

$$Z(n-p, k) - Z(n-p, k+1) = \begin{cases} Z(n, k) - Z(n, k+1), & \ell \neq k, \\ Z(n, k) - Z(n, k+1) - 1, & \ell = k, \end{cases}$$

while from (2.22) we get

$$d\sigma_{n-p, k}(\tau) = \begin{cases} d\sigma_{n, k}(\tau), & \ell \neq k, \\ \tau d\sigma_{n, k}(\tau), & \ell = k. \end{cases}$$

So we see that (3.11) holds in the case $\ell \leq p-1$. Similarly, if $\ell = p$, then $n-p \equiv 0 \pmod{p+1}$, so that, again by (2.30) and (2.22), we have

$$Z(n-p, k) - Z(n-p, k+1) = Z(n, k) - Z(n, k+1), \quad \ell = p,$$

and

$$\tau d\sigma_{n-p, k}(\tau) = d\sigma_{n, k}(\tau), \quad \ell = p.$$

□

Lemma 3.4. *Suppose that $n \equiv \ell \pmod{p+1}$, $0 \leq \ell \leq p$, and that $n = mp + k$ with $0 \leq k \leq p-1$. With the notation $\Delta_k := (a_k, b_k)$, we have*

$$\text{sign}(P_{n, k-1} P_{n, k+1}, \Delta_k) = \begin{cases} 1, & \text{if } k \text{ is even,} \\ -1, & \text{if } k \neq \ell, k \text{ odd,} \\ 1, & \text{if } k = \ell, k \text{ odd.} \end{cases} \quad (3.16)$$

Also, for every j in the range $0 \leq j \leq k$, we have

$$\text{sign}(H_{n, j}, \Delta_j) = \begin{cases} (-1)^j, & j \leq \ell, \\ 1, & \ell < j. \end{cases} \quad (3.17)$$

Proof. From (2.64), we know that for every $1 \leq j \leq p-1$,

$$\text{sign}(H_{n,j}, \Delta_j) = \begin{cases} (-1)^{1+(j+1)[Z(n,j-2)-Z(n,j)]} \text{sign}(H_{n,j-1}, \Delta_{j-1}), & j \leq \ell, \\ (-1)^{(j+1)[Z(n,j-2)-Z(n,j)]} \text{sign}(H_{n,j-1}, \Delta_{j-1}), & \ell < j, \end{cases} \quad (3.18)$$

which will allow us to recursively compute the sign of $H_{n,j}$.

From (2.51), we get that for all $j < k$,

$$Z(n, j) - Z(n, j+1) = \begin{cases} \lambda + 1, & \ell \leq j \text{ or } k < \ell, \\ \lambda, & j < \ell \leq k, \end{cases}$$

while if $k \leq j$, then

$$Z(n, j) - Z(n, j+1) = \begin{cases} \lambda, & \ell \leq k \text{ or } j < \ell, \\ \lambda + 1, & k < \ell \leq j. \end{cases}$$

Since

$$Z(n, j-2) - Z(n, j) = Z(n, j-2) - Z(n, j-1) + Z(n, j-1) - Z(n, j),$$

this implies that if $2 \leq j \leq k$, then

$$Z(n, j-2) - Z(n, j) = \begin{cases} 2(\lambda + 1), & \ell < j-1 \text{ or } k < \ell, \\ 2\lambda + 1, & \ell = j-1, \\ 2\lambda, & j-1 < \ell \leq k, \end{cases} \quad (3.19)$$

and if $j = k+1$, then

$$Z(n, k-1) - Z(n, k+1) = \begin{cases} 2\lambda + 1, & \ell \neq k, \\ 2\lambda, & \ell = k. \end{cases} \quad (3.20)$$

The validity of (3.16) for k even is trivial, since in such a case, $\Delta_k \subset (0, \infty)$ while the zeros of the monic polynomials $P_{n,k \pm 1}$ all lie in $\Delta_{k \pm 1} \subset (-\infty, 0)$.

If $k \geq 1$ is odd, then $\Delta_k \subset (-\infty, 0)$, $\Delta_{k-1}, \Delta_{k+1} \subset (0, \infty)$, so that

$$\text{sign}(P_{n,k-1} P_{n,k+1}, \Delta_k) = (-1)^{Z(n,k-1) - Z(n,k+1)}$$

and (3.16) for k odd follows from (3.20).

Now, directly from (3.18) we get

$$\text{sign}(H_{n,1}, \Delta_1) = \begin{cases} \text{sign}(H_{n,0}, \Delta_0), & \ell = 0, \\ -\text{sign}(H_{n,0}, \Delta_0), & \ell \geq 1, \end{cases} \quad (3.21)$$

while from (3.19) and (3.18), we obtain that for all $2 \leq j \leq k$,

$$\text{sign}(H_{n,j}, \Delta_j) = \begin{cases} -\text{sign}(H_{n,j-1}, \Delta_{j-1}), & k < \ell, \\ \text{sign}(H_{n,j-1}, \Delta_{j-1}), & \ell < j-1, \\ (-1)^{j+1} \text{sign}(H_{n,j-1}, \Delta_{j-1}), & \ell = j-1, \\ -\text{sign}(H_{n,j-1}, \Delta_{j-1}), & j \leq \ell \leq k. \end{cases} \quad (3.22)$$

This implies that

$$\text{sign}(H_{n,j}, \Delta_j) = -\text{sign}(H_{n,j-1}, \Delta_{j-1}), \quad 1 \leq j \leq \ell,$$

and iterating this relation we obtain (recall that $H_{n,0} \equiv 1$)

$$\text{sign}(H_{n,j}, \Delta_j) = (-1)^j, \quad 0 \leq j \leq \ell. \quad (3.23)$$

We now get from (3.22) and (3.23) that if $\ell < j \leq k$ and $j \geq 2$, then

$$\text{sign}(H_{n,j}, \Delta_j) = \text{sign}(H_{n,\ell+1}, \Delta_{\ell+1}) = (-1)^\ell \text{sign}(H_{n,\ell}, \Delta_\ell) = 1.$$

By (3.21), we see that this last relation also holds if $j = 1 > \ell = 0$, completing the proof of (3.17). \square

Theorem 3.5. *The coefficients a_n of the recurrence relation (3.1) are all positive, i.e., $a_n > 0$ for every $n \geq p$.*

Proof. It follows directly from (3.7)–(3.8) and Lemma 3.3 that for all $n \geq p$, with $n \equiv k \pmod p$, we have

$$\int_{a_k}^{b_k} \tau^{Z(n,k)-Z(n,k+1)} \psi_{n,k}(\tau) d\sigma_{n,k}(\tau) = a_n \int_{a_k}^{b_k} \tau^{Z(n-p,k)-Z(n-p,k+1)} \psi_{n-p,k}(\tau) d\sigma_{n-p,k}(\tau). \quad (3.24)$$

Since $\deg(P_{n,k}) = Z(n,k)$, we get from (2.56) that

$$\begin{aligned} \int_{a_k}^{b_k} \tau^{Z(n,k)-Z(n,k+1)} \psi_{n,k}(\tau) d\sigma_{n,k}(\tau) &= \int_{a_k}^{b_k} \tau^{Z(n,k)-Z(n,k+1)} P_{n,k+1}(\tau) \frac{\psi_{n,k}(\tau)}{P_{n,k+1}(\tau)} d\sigma_{n,k}(\tau) \\ &= \int_{a_k}^{b_k} P_{n,k}(\tau) \frac{\psi_{n,k}(\tau)}{P_{n,k+1}(\tau)} d\sigma_{n,k}(\tau) \\ &= \int_{a_k}^{b_k} P_{n,k}^2(\tau) \frac{H_{n,k}(\tau)}{P_{n,k-1}(\tau) P_{n,k+1}(\tau)} d\sigma_{n,k}(\tau). \end{aligned} \quad (3.25)$$

It follows from Lemma 3.4 and (2.22) that if $n = mp + k$, then

$$\text{sign} \left(\int_{a_k}^{b_k} P_{n,k}^2(\tau) \frac{H_{n,k}(\tau)}{P_{n,k-1}(\tau) P_{n,k+1}(\tau)} d\sigma_{n,k}(\tau) \right) = (-1)^k,$$

and since $n - p = (m - 1)p + k$, we conclude that the two integrals in (3.24) have the same sign, and thus $a_n > 0$. \square

Corollary 3.6. *The non-zero roots of the polynomials Q_n and Q_{n+1} interlace on Γ_0 for every $n \geq p + 1$, i.e., between two consecutive non-zero roots of Q_n there is exactly one non-zero root of Q_{n+1} and vice versa.*

Proof. This interlacing property is a consequence of (3.1)–(3.2) and the positivity of the recurrence coefficients, as it was shown in Theorem 2.2 from [13]. \square

Figure 1 illustrates the interlacing property. We remark that for every $k = 1, \dots, p - 1$, the zeros of the polynomials $P_{n,k}$ and $P_{n+1,k}$ also interlace on $[a_k, b_k]$. This property will be proved in a subsequent work.

4 Normalization

In this section we introduce a convenient normalization of the polynomials $P_{n,k}$ and the functions $H_{n,k}$.

It follows from the definition of the functions $H_{n,k}$ and the polynomials $P_{n,k}$ that the measures

$$\frac{H_{n,k}(\tau) d\sigma_{n,k}(\tau)}{P_{n,k-1}(\tau) P_{n,k+1}(\tau)}, \quad 0 \leq k \leq p - 1,$$

have constant sign on the interval $[a_k, b_k]$. We then denote by

$$\frac{|H_{n,k}(\tau)| d|\sigma_{n,k}|(\tau)}{|P_{n,k-1}(\tau) P_{n,k+1}(\tau)|}$$

the positive normalization of this measure and we have

$$\int_{a_k}^{b_k} P_{n,k}(\tau) \tau^s \frac{|H_{n,k}(\tau)| d|\sigma_{n,k}|(\tau)}{|P_{n,k-1}(\tau) P_{n,k+1}(\tau)|} = 0, \quad s = 0, \dots, Z(n, k) - 1, \quad k = 0, \dots, p-1. \quad (4.1)$$

Let

$$K_{n,-1} := 1, \quad K_{n,p} := 1, \quad (4.2)$$

$$K_{n,k} := \left(\int_{a_k}^{b_k} P_{n,k}^2(\tau) \frac{|H_{n,k}(\tau)| d|\sigma_{n,k}|(\tau)}{|P_{n,k-1}(\tau) P_{n,k+1}(\tau)|} \right)^{-1/2}, \quad k = 0, \dots, p-1, \quad (4.3)$$

and we also define the constants

$$\kappa_{n,k} := \frac{K_{n,k}}{K_{n,k-1}}, \quad k = 0, \dots, p. \quad (4.4)$$

Definition 4.1. For $k = 0, \dots, p$, we define

$$p_{n,k} := \kappa_{n,k} P_{n,k}, \quad (4.5)$$

$$h_{n,k} := K_{n,k-1}^2 H_{n,k}, \quad (4.6)$$

where the constants $\kappa_{n,k}$ and $K_{n,k}$ are given in (4.4) and (4.2)–(4.3), respectively.

We will denote by $\nu_{n,k}$ the measure on $[a_k, b_k]$ given by

$$d\nu_{n,k}(\tau) := \frac{h_{n,k}(\tau) d\sigma_{n,k}(\tau)}{P_{n,k-1}(\tau) P_{n,k+1}(\tau)}, \quad k = 0, \dots, p-1.$$

Again this measure has constant sign in $[a_k, b_k]$, and we will denote by $\varepsilon_{n,k}$ its sign and by $|\nu_{n,k}|$ its positive normalization, hence

$$d|\nu_{n,k}|(\tau) = \frac{|h_{n,k}(\tau)| d|\sigma_{n,k}|(\tau)}{|P_{n,k-1}(\tau) P_{n,k+1}(\tau)|} = \varepsilon_{n,k} \frac{h_{n,k}(\tau) d\sigma_{n,k}(\tau)}{P_{n,k-1}(\tau) P_{n,k+1}(\tau)}. \quad (4.7)$$

Proposition 4.2. For each $k = 0, \dots, p-1$, the polynomial $p_{n,k}$ defined in (4.5) satisfies the following:

$$\int_{a_k}^{b_k} p_{n,k}(\tau) \tau^s d|\nu_{n,k}|(\tau) = 0, \quad s = 0, \dots, Z(n, k) - 1, \quad (4.8)$$

$$\int_{a_k}^{b_k} p_{n,k}^2(\tau) d|\nu_{n,k}|(\tau) = 1, \quad (4.9)$$

that is, $p_{n,k}$ is the orthonormal polynomial of degree $Z(n, k)$ with respect to the positive measure $|\nu_{n,k}|$.

For each $k = 1, \dots, p$, the function $h_{n,k}$ defined in (4.6) satisfies

$$h_{n,k}(z) = \begin{cases} \varepsilon_{n,k-1} \int_{a_{k-1}}^{b_{k-1}} \frac{p_{n,k-1}^2(\tau)}{z-\tau} d|\nu_{n,k-1}|(\tau) & \text{if } k \leq \ell, \\ \varepsilon_{n,k-1} z \int_{a_{k-1}}^{b_{k-1}} \frac{p_{n,k-1}^2(\tau)}{z-\tau} d|\nu_{n,k-1}|(\tau) & \text{if } k > \ell. \end{cases} \quad (4.10)$$

Proof. The orthogonality conditions (4.8) are obvious in view of (4.1). The formulas (4.9) and (4.10) follow immediately from (4.5)–(4.6), (4.4), (4.2)–(4.3) and (2.60). \square

5 Zero asymptotic distribution

5.1 Definitions and results

In this section we investigate the zero asymptotic distribution of the polynomials Q_n . This distribution will be described in terms of a vector equilibrium problem for logarithmic potentials. Before describing this problem, let us introduce some definitions and notations.

Let E_k , $k = 0, \dots, p-1$ be a system of compact subsets of the real line satisfying

$$E_k \cap E_{k+1} = \emptyset, \quad k = 0, \dots, p-2. \quad (5.1)$$

We assume that

$$\text{cap}(E_k) > 0, \quad k = 0, \dots, p-1, \quad (5.2)$$

where $\text{cap}(E)$ denotes the logarithmic capacity of a compact set E . A vector measure

$$\vec{\nu} = (\nu_0, \nu_1, \dots, \nu_{p-1})$$

is called *admissible* if

- 1) ν_k is a positive Borel measure supported on E_k for all $k = 0, \dots, p-1$;
- 2) ν_k has total mass $\|\nu_k\| = 1 - \frac{k}{p}$ for all $k = 0, \dots, p-1$.

We denote by \mathcal{M} the class of all admissible vector measures.

Given a pair of compactly supported measures ν_1, ν_2 , let $I(\nu_1)$ and $I(\nu_1, \nu_2)$ denote, respectively, the logarithmic energy of ν_1 and the mutual logarithmic energy of ν_1 and ν_2 defined by

$$I(\nu_1) = \iint \log \frac{1}{|x-y|} d\nu_1(x) d\nu_1(y), \quad I(\nu_1, \nu_2) = \iint \log \frac{1}{|x-y|} d\nu_1(x) d\nu_2(y).$$

On the class of admissible vector measures $\vec{\nu} = (\nu_0, \dots, \nu_{p-1})$ we consider the energy functional J defined by

$$J(\vec{\nu}) := \sum_{k=0}^{p-1} I(\nu_k) - \sum_{k=0}^{p-2} I(\nu_k, \nu_{k+1}). \quad (5.3)$$

Observe that J is well-defined and $J(\vec{\nu}) \in (-\infty, +\infty]$ for all $\vec{\nu} \in \mathcal{M}$. This type of energy interaction is typical in the study of Nikishin systems on the real line.

The vector equilibrium problem that is relevant in this work is the problem of finding an extremal vector measure $\vec{\mu} \in \mathcal{M}$ that satisfies

$$J(\vec{\mu}) = \inf_{\vec{\nu} \in \mathcal{M}} J(\vec{\nu}) < \infty. \quad (5.4)$$

Such a measure exists and is unique, see [11] for a proof of this fact and several other important results on logarithmic vector equilibrium problems in the complex plane. The extremal measure $\vec{\mu}$ is the vector *equilibrium measure*.

The vector equilibrium measure can be characterized in terms of certain equilibrium conditions that we describe next. Given a vector measure $\vec{\nu} = (\nu_0, \dots, \nu_{p-1}) \in \mathcal{M}$, we consider the combined potentials $W_k^{\vec{\nu}}$ defined by

$$W_k^{\vec{\nu}}(z) = U^{\nu_k}(z) - \frac{1}{2}U^{\nu_{k-1}}(z) - \frac{1}{2}U^{\nu_{k+1}}(z), \quad k = 0, \dots, p-1, \quad (5.5)$$

where U^ν denotes the logarithmic potential associated with ν , i.e.,

$$U^\nu(z) = \int \log \frac{1}{|z-t|} d\nu(t),$$

and in (5.5) we understand $U^{\nu_{-1}} \equiv 0$, $U^{\nu_p} \equiv 0$. The following result is an adaptation of a well-known result in the theory of logarithmic vector equilibrium problems, see [11].

Lemma 5.1. *Let $\vec{\mu} = (\mu_0, \dots, \mu_{p-1}) \in \mathcal{M}$ be the vector equilibrium measure satisfying (5.4). Then there exist finite constants $\{w_k\}_{k=0}^{p-1}$ such that for every $k = 0, \dots, p-1$ the following conditions hold:*

$$W_k^{\vec{\mu}}(x) \leq w_k, \quad \text{for all } x \in \text{supp}(\mu_k), \quad (5.6)$$

$$W_k^{\vec{\mu}}(x) \geq w_k, \quad \text{for q.e. } x \in E_k. \quad (5.7)$$

Conversely, if $\vec{\mu} \in \mathcal{M}$ and there exist constants $\{w_k\}_{k=0}^{p-1}$ such that (5.6) and (5.7) hold for every $k = 0, \dots, p-1$, then $\vec{\mu}$ is the vector equilibrium measure satisfying (5.4).

Let $E \subset \mathbb{C}$ be a compact set, let $\{\nu_n\}_n$ be a sequence of finite positive measures supported on E , and let ν be another finite positive measure on E . We write

$$\nu_n \xrightarrow[n \rightarrow \infty]{*} \nu$$

if for every $f \in C(E)$,

$$\lim_{n \rightarrow \infty} \int f d\nu_n = \int f d\nu,$$

i.e., when the sequence of measures converges to ν in the weak-star topology. Given a polynomial P of degree $n \geq 1$, we denote the associated normalized zero counting measure by

$$\mu_P = \frac{1}{n} \sum_{P(z)=0} \delta_z,$$

where δ_z is the Dirac mass at z (in the sum the zeros are repeated according to their multiplicity).

The weak asymptotic result that we present in this paper is obtained under mild assumptions on the measures σ generating the Nikishin system. One of these assumptions is the so-called regularity of the measures in the sense of Stahl and Totik. A measure σ is said to be in the class **Reg** if

$$\lim_{n \rightarrow \infty} \|\pi_n\|_{L^2(\sigma)}^{1/n} = \text{cap}(\text{supp}(\sigma)),$$

where π_n denotes the n th monic orthogonal polynomial associated with the measure σ .

These monic polynomials have a very important extremal property, namely, among all monic polynomials of degree n , they have the smallest $L_2(\sigma)$ -norm:

$$\|\pi_n\|_{L^2(\sigma)} = \min_{P(z)=z^n+\dots} \|P\|_{L^2(\sigma)}.$$

We refer the reader to [15] for a detailed analysis of the orthogonal polynomials associated with measures in the class **Reg**. It is well-known that the regularity assumption is indeed a mild condition. For instance, measures σ supported on a compact interval $I \subset \mathbb{R}$ on which $\sigma'(x) > 0$ a.e. are regular.

Let $E \subset \mathbb{C}$ be a compact set with $\text{cap}(E) > 0$ and let φ be a continuous function on E . Recall that the equilibrium measure $\bar{\mu}$ in the presence of the external field φ is the unique probability measure that minimizes the energy functional $I(\mu) + \int \varphi d\mu$ among all probability measures on E , cf. [14]. The equilibrium measure $\bar{\mu}$ satisfies

$$U^{\bar{\mu}}(z) + \varphi(z) \begin{cases} \leq w, & \text{for all } z \in \text{supp}(\bar{\mu}), \\ \geq w, & \text{for q.e. } z \in E, \end{cases} \quad (5.8)$$

for some constant w (called the equilibrium constant). These equilibrium conditions also characterize the equilibrium measure, and we emphasize that if E is regular with respect to the Dirichlet problem, then in (5.8) the first inequality can be replaced by an equality and the second inequality holds for all $z \in E$.

We will need the following auxiliary result concerning the zero asymptotic distribution of a sequence of orthogonal polynomials with respect to varying measures.

Lemma 5.2. Let $\sigma \in \mathbf{Reg}$, $E = \text{supp}(\sigma) \subset \mathbb{R}$, where E is regular with respect to the Dirichlet problem. Let $\{\phi_l\}$, $l \in \Lambda \subset \mathbb{Z}_+$, be a sequence of positive continuous functions on E such that

$$\lim_{l \in \Lambda} \frac{1}{2l} \log \frac{1}{|\phi_l(x)|} = \varphi(x) > -\infty,$$

uniformly on E . Let q_l , $l \in \Lambda$, be a sequence of monic polynomials such that $\deg q_l = l$ and

$$\int x^k q_l(x) \phi_l(x) d\sigma(x) = 0, \quad k = 0, \dots, l-1.$$

Then

$$\mu_{q_l} \xrightarrow[l \in \Lambda]{*} \bar{\mu}$$

and

$$\lim_{l \in \Lambda} \left(\int |q_l(x)|^2 \phi_l(x) d\sigma(x) \right)^{1/2l} = e^{-w},$$

where $\bar{\mu}$ and w are the equilibrium measure and equilibrium constant in the presence of the external field φ on E .

The above result was proved in [4]. It is a generalization of a result of Gonchar and Rakhmanov [6] obtained under the more restrictive assumption that $\text{supp}(\sigma)$ is an interval on which $\sigma' > 0$ a.e.

In the following asymptotic results, the measures μ_k are the components of the vector equilibrium measure $\vec{\mu} = (\mu_0, \dots, \mu_{p-1})$ that minimizes the energy functional (5.3) on the space \mathcal{M} of all admissible vector measures supported on $E_k = \text{supp}(\sigma_k^*)$, $k = 0, \dots, p-1$, and the constants w_k are the equilibrium constants satisfying the variational conditions (5.6)-(5.7).

Theorem 5.3. Let $(s_0, \dots, s_{p-1}) = \mathcal{N}(\sigma_0, \dots, \sigma_{p-1})$ be the Nikishin system generated by the measures $\sigma_0, \dots, \sigma_{p-1}$. Assume that for each $k = 0, \dots, p-1$, the measure σ_k satisfies $\sigma_k \in \mathbf{Reg}$ and $\text{supp}(\sigma_k)$ is regular for the Dirichlet problem. Then, for each $k = 0, \dots, p-1$, we have $\text{supp}(\mu_k) = \text{supp}(\sigma_k^*)$ and

$$\mu_{P_{n,k}} \xrightarrow[n \rightarrow \infty]{*} \frac{p}{p-k} \mu_k. \quad (5.9)$$

The results that we state in what follows are obtained under the same assumptions of Theorem 5.3.

Theorem 5.4. For every $k = 0, \dots, p$,

$$\lim_{n \rightarrow \infty} |\psi_{n,k}(z)|^{1/Z(n,0)} = e^{-U^{\mu_k}(z) + U^{\mu_{k-1}}(z) - 2 \sum_{j=0}^{k-1} w_j} \quad (5.10)$$

uniformly on compact subsets of $\mathbb{C} \setminus ([a_{k-1}, b_{k-1}] \cup [a_k, b_k] \cup \{0\})$. In (5.10) we understand $U^{\mu_{-1}}, U^{\mu_p} \equiv 0$.

Theorem 5.5. For every $k = 0, \dots, p-1$,

$$\lim_{m \rightarrow \infty} \left(\prod_{j=1}^m a_{pj+k} \right)^{1/m} = e^{-\frac{2p}{p+1} \sum_{j=0}^k w_j}.$$

We now state the corresponding asymptotic results on the stars Γ_k . For each $k = 0, \dots, p-1$, let $\tilde{\mu}_k$ be the unique rotationally symmetric measure supported on Γ_k such that for every Borel set $E \subset [a_k, b_k]$,

$$\tilde{\mu}_k(\{z : z^{p+1} \in E\}) = \mu_k(E). \quad (5.11)$$

Let $\omega_{k,j}$, $j = 0, \dots, p$, be the $p+1$ distinct roots of the equation $z^{p+1} = (-1)^k$, numbered as usual in such a way that $0 \leq \arg \omega_{k,j} < \arg \omega_{k,j+1} < 2\pi$. Then we can write $\Gamma_k = \cup_{j=0}^p \Gamma_{k,j}$, with

$$\Gamma_{k,j} = \{z : z^{p+1} \in [a_k, b_k], \ z/\omega_{k,j} \geq 0\}.$$

Then, for every Borel set $F \subset \Gamma_{k,j}$,

$$\tilde{\mu}_k|_{\Gamma_{k,j}}(F) = \frac{1}{p+1} \mu_k(\{z^{p+1} : z \in F\}).$$

Corollary 5.6. *For the zero counting measures μ_{Q_n} of the multi-orthogonal polynomials Q_n , we have*

$$\mu_{Q_n} \xrightarrow[n \rightarrow \infty]{*} \tilde{\mu}_0. \quad (5.12)$$

Corollary 5.7. *For every $k = 0, \dots, p$,*

$$\lim_{n \rightarrow \infty} |\Psi_{n,k}(z)|^{1/n} = e^{-U^{\tilde{\mu}_k}(z) + U^{\tilde{\mu}_{k-1}}(z) - \frac{2}{p+1} \sum_{j=0}^{k-1} w_j} \quad (5.13)$$

uniformly on compact subsets of $\mathbb{C} \setminus (\Gamma_k \cup \Gamma_{k-1} \cup \{0\})$. In (5.13) we understand $U^{\tilde{\mu}_{-1}}, U^{\tilde{\mu}_p} \equiv 0$.

The proofs of these asymptotic results make use of a few auxiliary lemmas that we present in the next section.

5.2 Some auxiliary lemmas

Lemma 5.8. *Let $Z(n, k)$, $k = 0, \dots, p-1$, be the constants given in (2.36). Then we have*

$$\lim_{n \rightarrow \infty} \frac{Z(n, k)}{Z(n, k-1)} = \frac{p-k}{p-k+1}, \quad k = 1, \dots, p-1. \quad (5.14)$$

Proof. Follows immediately from (2.48). \square

Lemma 5.9. *Let σ_j be a positive, rotationally symmetric measure on the star $\Gamma_j = \{z \in \mathbb{C} : z^{p+1} \in [a_j, b_j]\}$, for some $j = 0, \dots, p-1$, and suppose that $\sigma_j \in \mathbf{Reg}$. Then the measures $d\sigma_j^*(\tau)$ and $|\tau| d\sigma_j^*(\tau)$ on $[a_j, b_j]$, where σ_j^* is defined in (2.8), are also in the class \mathbf{Reg} .*

Proof. We begin by observing that, since

$$\text{supp}(\sigma_j) = \{z : z^{p+1} \in \text{supp}(\sigma_j^*)\}$$

we have (see [12, Thm. 5.2.5])

$$[\text{cap}(\text{supp}(\sigma_j))]^{p+1} = \text{cap}(\text{supp}(\sigma_j^*)).$$

Let π_n be the n th monic orthogonal polynomial associated with the measure σ_j . Then $\sigma_j \in \mathbf{Reg}$ means that

$$\lim_{n \rightarrow \infty} \|\pi_n\|_{L_2(\sigma_j)}^{1/n} = \text{cap}(\text{supp}(\sigma_j)). \quad (5.15)$$

The polynomial π_n is the monic polynomial of degree n that satisfies the orthogonality conditions

$$\int_{\Gamma_j} \pi_n(z) \overline{z^k} d\sigma_j(z) = 0, \quad k = 0, \dots, n-1.$$

By the rotational symmetry of σ_j , the monic polynomial $\omega^{-n} \pi_n(\omega z)$ (where $\omega = e^{\frac{2\pi i}{p+1}}$) satisfies the same orthogonality conditions, and therefore, for every integer $m \geq 0$,

$$\pi_{m(p+1)}(z) = L_m(z^{p+1})$$

for some monic polynomial L_m of degree m .

For $k = 0, \dots, m-1$,

$$\begin{aligned} 0 &= \int_{\Gamma_j} \pi_{m(p+1)}(z) \overline{z^{k(p+1)}} d\sigma_j(z) = \int_{\Gamma_j} L_m(z^{p+1}) \overline{z^{k(p+1)}} d\sigma_j(z) \\ &= \int_{a_j}^{b_j} L_m(\tau) \overline{\tau^k} d\sigma_j^*(\tau). \end{aligned}$$

Hence the sequence $(L_m)_{m=0}^\infty$ is the sequence of monic orthogonal polynomials associated with the measure σ_j^* .

Similarly, we have

$$\begin{aligned} \|\pi_{m(p+1)}\|_{L^2(\sigma_j)}^2 &= \int_{\Gamma_j} |\pi_{m(p+1)}(z)|^2 d\sigma_j(z) = \int_{\Gamma_j} |L_m(z^{p+1})|^2 d\sigma_j(z) \\ &= \int_{a_j}^{b_j} |L_m(\tau)|^2 d\sigma_j^*(\tau) = \|L_m\|_{L^2(\sigma_j^*)}^2, \end{aligned}$$

and therefore from (5.15) it follows that

$$\lim_{m \rightarrow \infty} \|L_m\|_{L^2(\sigma_j^*)}^{1/m} = \lim_{m \rightarrow \infty} \|\pi_{m(p+1)}\|_{L^2(\sigma_j)}^{1/m} = [\text{cap}(\text{supp}(\sigma_j))]^{p+1} = \text{cap}(\text{supp}(\sigma_j^*)).$$

This proves that $\sigma_j^* \in \mathbf{Reg}$.

Now that we know that σ_j^* is regular, we want to conclude that the measure $d\lambda(\tau) := |\tau| d\sigma_j^*(\tau)$ is also regular. Let (l_n) be the sequence of monic orthogonal polynomials associated with the measure λ , and let (L_n) be the corresponding sequence for σ_j^* . Obviously, $\text{supp}(\sigma_j^*) = \text{supp}(\lambda)$, and so the regularity of λ is equivalent to showing that

$$\lim_{n \rightarrow \infty} \|l_n\|_{L^2(\lambda)}^{1/n} = \lim_{n \rightarrow \infty} \|L_n\|_{L^2(\sigma_j^*)}^{1/n}.$$

Without loss of generality, we assume that $0 \leq a_j \leq b_j$. Then, by the extremality property of the monic orthogonal polynomials, we have

$$b_j^{-1} \|L_{n+1}\|_{L^2(\sigma_j^*)}^2 \leq b_j^{-1} \int_{a_j}^{b_j} |l_n(\tau)|^2 \tau^2 d\sigma_j^*(\tau) \leq \|l_n\|_{L^2(\lambda)}^2 \leq \int_{a_j}^{b_j} |L_n|^2 d\lambda \leq b_j \|L_n\|_{L^2(\sigma_j^*)}^2.$$

Taking n th roots and letting $n \rightarrow \infty$ in this chain of inequalities yields the desired result. \square

5.3 Proof of Theorem 5.3

Let $\Lambda \subset \mathbb{N}$ be a sequence of integers such that for every $k = 0, \dots, p-1$,

$$c_k \mu_{P_{n,k}} \xrightarrow[n \in \Lambda]{*} \mu_k, \quad c_k := 1 - \frac{k}{p}, \quad (5.16)$$

for some positive measures μ_k on $[a_k, b_k]$. Our goal is to show that the vector measure $\vec{\mu} = (\mu_0, \dots, \mu_{p-1})$ is the unique equilibrium measure satisfying (5.4). This implies that for each $k = 0, \dots, p-1$, the sequence of measures $(\mu_{P_{n,k}})_n$ has a unique limit point in the weak-star topology. By the compactness of the unit ball in the space of Borel positive measures with the weak-star topology, we obtain that the limits hold.

Note that μ_k has mass c_k . Then we have

$$\lim_{n \in \Lambda} \frac{c_k}{Z(n,k)} \log |P_{n,k}(z)| = -U^{\mu_k}(z), \quad k = 0, \dots, p-1, \quad (5.17)$$

uniformly on compact subsets of $\mathbb{C} \setminus [a_k, b_k]$.

Observe that in fact $\text{supp}(\mu_k) \subset E_k := \text{supp}(\sigma_k^*)$ for every $k = 0, \dots, p-1$. This follows immediately from Corollary 2.22.

Consider first the sequence $(P_{n,0})$, $n \in \Lambda$. According to (2.59) and (2.22), we have the orthogonality conditions

$$\int_{a_0}^{b_0} P_{n,0}(\tau) \tau^s \frac{d|\sigma_{n,0}|(\tau)}{|P_{n,1}(\tau)|} = 0, \quad s = 0, \dots, Z(n,0) - 1,$$

where

$$d\sigma_{n,0}(\tau) = \begin{cases} d\sigma_0^*(\tau) & \text{if } \ell(n) = 0, \\ \tau d\sigma_0^*(\tau) & \text{if } \ell(n) > 0. \end{cases}$$

In virtue of (5.17) and (5.14) we have

$$\lim_{n \in \Lambda} \frac{\log |P_{n,1}(\tau)|}{2Z(n,0)} = \lim_{n \in \Lambda} \frac{Z(n,1)}{Z(n,0)} \frac{\log |P_{n,1}(\tau)|}{2Z(n,1)} = -\frac{1}{2} U^{\mu_1}(\tau)$$

uniformly on $E_0 = \text{supp}(\sigma_0^*)$. Now we can apply Lemma 5.2 to the sequence $(P_{n,0})$, identifying ϕ_l with the weight $1/|P_{n,1}|$ and φ with the function $-(1/2)U^{\mu_1}$. By Lemma 5.9, the measures σ_0^* and $|\tau d\sigma_0^*(\tau)|$ are regular. The regularity of $\text{supp}(\sigma_0)$ with respect to the Dirichlet problem also implies the regularity of $\text{supp}(\sigma_0^*)$ with respect to the Dirichlet problem. Therefore, we obtain from Lemma 5.2 that

$$\mu_{P_{n,0}} = \frac{1}{Z(n,0)} \sum_{P_{n,0}(x)=0} \delta_x \xrightarrow[n \in \Lambda]{*} \bar{\mu}_0, \quad (5.18)$$

and

$$\lim_{n \in \Lambda} \left(\int P_{n,0}^2(\tau) \frac{d|\sigma_{n,0}|(\tau)}{|P_{n,1}(\tau)|} \right)^{\frac{1}{2Z(n,0)}} = e^{-\bar{w}_0},$$

where $\bar{\mu}_0$ and \bar{w}_0 are the equilibrium measure and equilibrium constant, respectively, in the presence of the external field $\varphi = -\frac{1}{2}U^{\mu_1}$ on E_0 .

Consequently, from (5.16) and (5.18) we deduce that $\mu_0 = \bar{\mu}_0$ and so μ_0 satisfies

$$U^{\mu_0}(x) - \frac{1}{2}U^{\mu_1}(x) \begin{cases} = \bar{w}_0, & \text{for } x \in \text{supp}(\mu_0), \\ \geq \bar{w}_0, & \text{for } x \in E_0 \setminus \text{supp}(\mu_0). \end{cases} \quad (5.19)$$

Now assume that $1 \leq k \leq p-1$. We have the orthogonality conditions

$$\int_{a_k}^{b_k} P_{n,k}(\tau) \tau^s \frac{|h_{n,k}(\tau)| d|\sigma_{n,k}|(\tau)}{|P_{n,k-1}(\tau) P_{n,k+1}(\tau)|} = 0, \quad s = 0, \dots, Z(n,k) - 1, \quad (5.20)$$

where

$$d\sigma_{n,k}(\tau) = \begin{cases} d\sigma_k^*(\tau), & \text{if } k \geq \ell(n), \\ \tau d\sigma_k^*(\tau) & \text{if } k < \ell(n), \end{cases}$$

and the function $h_{n,k}$ is defined in (4.6). We consider the expression

$$\frac{1}{2Z(n,k)} \log \left(\frac{|P_{n,k-1}(\tau)| |P_{n,k+1}(\tau)|}{|h_{n,k}(\tau)|} \right) = \frac{\log |P_{n,k-1}(\tau)| + \log |P_{n,k+1}(\tau)| - \log |h_{n,k}(\tau)|}{2Z(n,k)} \quad (5.21)$$

associated with the orthogonality measure in (5.20). Recall also that $P_{n,p} \equiv 1$. Applying (5.17) and (5.14) we obtain

$$\lim_{n \in \Lambda} \frac{\log |P_{n,k-1}(\tau)|}{2Z(n,k)} = \lim_{n \in \Lambda} \frac{Z(n,k-1)}{Z(n,k)} \frac{\log |P_{n,k-1}(\tau)|}{2Z(n,k-1)} = -\frac{p}{2(p-k)} U^{\mu_{k-1}}(\tau), \quad (5.22)$$

$$\lim_{n \in \Lambda} \frac{\log |P_{n,k+1}(\tau)|}{2Z(n,k)} = \lim_{n \in \Lambda} \frac{Z(n,k+1)}{Z(n,k)} \frac{\log |P_{n,k+1}(\tau)|}{2Z(n,k+1)} = -\frac{p}{2(p-k)} U^{\mu_{k+1}}(\tau), \quad (5.23)$$

uniformly on $E_k = \text{supp}(\sigma_k^*)$. From (4.10) we see that for $z \in [a_k, b_k]$,

$$|h_{n,k}(z)| = \begin{cases} \int \frac{p_{n,k-1}^2(\tau)}{|z-\tau|} d|\nu_{n,k-1}|(\tau), & \text{if } k \leq \ell(n), \\ |z| \int \frac{p_{n,k-1}^2(\tau)}{|z-\tau|} d|\nu_{n,k-1}|(\tau), & \text{if } k > \ell(n), \end{cases} \quad (5.24)$$

and (4.9) shows that there exist constants $d_k > 0$, $D_k > 0$, independent of n , such that

$$d_k \leq \int \frac{p_{n,k-1}^2(\tau)}{|z-\tau|} d|\nu_{n,k-1}|(\tau) \leq D_k, \quad z \in [a_k, b_k]. \quad (5.25)$$

It then follows from (5.24) and (5.25) that

$$\lim_{n \in \Lambda} \frac{\log |h_{n,k}(\tau)|}{2Z(n,k)} = 0, \quad (5.26)$$

uniformly on $E_k = \text{supp}(\sigma_k^*)$. Here an observation needs to be made concerning the possibility that $0 \in E_k$. If this happens and $\ell(n) < k$, then $|h_{n,k}(z)|$ has a factor $|z|$ (see (5.24)) which may destroy (5.26) at $\tau = 0$. To avoid this, one could, if necessary, take the limit as $n \rightarrow \infty$ along a subsequence $\Lambda' \subset \Lambda$ such that $\ell(n) < k$ for all $n \in \Lambda'$ and incorporate the factor $|z|$ to the measure $|\sigma_{n,k}|$ in (5.20). So we may assume that (5.26) holds in any case. As a result of (5.22), (5.23) and (5.26) we have the convergence of (5.21) to

$$\lim_{n \in \Lambda} \frac{1}{2Z(n,k)} \log \left(\frac{|P_{n,k-1}(\tau)| |P_{n,k+1}(\tau)|}{|h_{n,k}(\tau)|} \right) = -\frac{p}{2(p-k)} (U^{\mu_{k-1}}(\tau) + U^{\mu_{k+1}}(\tau)),$$

uniformly on E_k .

As before, applying Lemmas 5.9 and 5.2 we obtain that

$$\mu_{P_{n,k}} \xrightarrow[n \rightarrow \Lambda]{*} \bar{\mu}_k \quad (5.27)$$

and

$$\lim_{n \in \Lambda} \left(\int_{a_k}^{b_k} P_{n,k}^2(\tau) \frac{|h_{n,k}(\tau)| |d|\sigma_{n,k}|(\tau)|}{|P_{n,k-1}(\tau)| |P_{n,k+1}(\tau)|} \right)^{\frac{1}{2Z(n,k)}} = e^{-\bar{w}_k},$$

where $\bar{\mu}_k$ and \bar{w}_k are the equilibrium measure and equilibrium constant, respectively, in the presence of the external field

$$\varphi(z) = -\frac{p}{2(p-k)} (U^{\mu_{k-1}}(z) + U^{\mu_{k+1}}(z))$$

on E_k . Therefore we have

$$U^{\bar{\mu}_k}(x) - \frac{p}{2(p-k)} U^{\mu_{k-1}}(x) - \frac{p}{2(p-k)} U^{\mu_{k+1}}(x) \begin{cases} = \bar{w}_k, & \text{for } x \in \text{supp}(\bar{\mu}_k), \\ \geq \bar{w}_k, & \text{for } x \in E_k \setminus \text{supp}(\bar{\mu}_k). \end{cases}$$

It follows from (5.16) and (5.27) that $\mu_k = \frac{p-k}{p} \bar{\mu}_k$ and so we have for every $k = 1, \dots, p-1$ that

$$U^{\mu_k}(x) - \frac{1}{2} U^{\mu_{k-1}}(x) - \frac{1}{2} U^{\mu_{k+1}}(x) \begin{cases} = \frac{p-k}{p} \bar{w}_k, & \text{for } x \in \text{supp}(\mu_k), \\ \geq \frac{p-k}{p} \bar{w}_k, & \text{for } x \in E_k \setminus \text{supp}(\mu_k). \end{cases} \quad (5.28)$$

where we understand $U^{\mu_p} \equiv 0$.

Finally, let $w_k := c_k \bar{w}_k$, $k = 0, \dots, p-1$. Then (5.19) and (5.28) show that the vector measure $\vec{\mu} = (\mu_0, \dots, \mu_{p-1}) \in \mathcal{M}$ satisfies the variational conditions (5.6)-(5.7) for every $k = 0, \dots, p-1$ (cf. (5.5)). Therefore, by Lemma 5.1, we obtain that $\vec{\mu}$ is the unique equilibrium measure satisfying (5.4).

Finally, from (5.19), (5.28) and (6.12) we deduce that $E_k \setminus \text{supp}(\mu_k) = \emptyset$ for all k , hence $\text{supp}(\mu_k) = \text{supp}(\sigma_k^*)$ for all k .

5.4 Proof of Theorem 5.4

From (2.58) and (4.6), we see that

$$|\psi_{n,k}(z)| = \frac{K_{n,k-1}^{-2} |h_{n,k}(z)| |P_{n,k}(z)|}{|P_{n,k-1}(z)|}, \quad 1 \leq k \leq p. \quad (5.29)$$

Thus, we seek to establish the asymptotic behavior of each of the factors in the right-hand side of (5.29).

Let us set

$$f_{n,k}(z) := \begin{cases} h_{n,k}(z), & \ell(n) < k, \\ z h_{n,k}(z), & \ell(n) \geq k. \end{cases}$$

From (4.9)-(4.10), we see that all functions in the family $\{f_{n,k}\}_n$ are analytic in $U := \overline{\mathbb{C}} \setminus ([a_{k-1}, b_{k-1}] \cup \{0\})$, and for every closed subset E of U , we have $\sup_{n \geq 0} \max_{z \in E} |f_{n,k}(z)| < \infty$. By Montel's theorem, $\{f_{n,k}\}_n$ is a normal family in U . Since no $f_{n,k}$ vanishes in U , and particularly, $|f_{n,k}(\infty)| = 1$, Hurwitz's theorem tells us that every normal limit point of $\{f_{n,k}\}_n$ is zero-free in U , which, in view of the normality of the family, implies that for every closed subset $E \subset U$, $\inf_{n \geq 0} \min_{z \in E} |f_{n,k}(z)| > 0$. Therefore, as $\lim_{n \rightarrow \infty} Z(n, 0) = \infty$, we have

$$\lim_{n \rightarrow \infty} |h_{n,k}(z)|^{1/Z(n,0)} = 1 \quad (5.30)$$

locally uniformly on $\mathbb{C} \setminus ([a_{k-1}, b_{k-1}] \cup \{0\})$.

It came out in the proof of Theorem 5.3 that

$$\lim_{n \rightarrow \infty} \left(\int_{a_k}^{b_k} P_{n,k}^2(\tau) \frac{|h_{n,k}(\tau)| d|\sigma_{n,k}|(\tau)}{|P_{n,k-1}(\tau) P_{n,k+1}(\tau)|} \right)^{\frac{1}{2Z(n,k)}} = e^{-\frac{w_k}{c_k}}, \quad k = 0, \dots, p-1,$$

where the constants w_k are the equilibrium constants satisfying the variational conditions (5.6)-(5.7). From this and the equalities (4.3) and (4.6), we obtain

$$\lim_{n \rightarrow \infty} \left[\frac{K_{n,k}}{K_{n,k-1}} \right]^{1/Z(n,k)} = e^{w_k/c_k}, \quad k = 0, \dots, p-1.$$

Since $K_{n,-1} = 1$ and $c_0 = 1$, this yields

$$\lim_{n \rightarrow \infty} K_{n,0}^{1/Z(n,0)} = e^{w_0}.$$

More generally,

$$\lim_{n \rightarrow \infty} K_{n,k}^{1/Z(n,0)} = e^{\sum_{j=0}^k w_j}, \quad k = 0, \dots, p-1, \quad (5.31)$$

which easily follows by mathematical induction. Indeed,

$$\lim_{n \rightarrow \infty} \frac{Z(n,k)}{Z(n,0)} = \lim_{n \rightarrow \infty} \prod_{j=1}^k \frac{Z(n,j)}{Z(n,j-1)} = \prod_{j=1}^k \frac{p-j}{p-(j-1)} = \frac{p-k}{p} = c_k, \quad 0 \leq k \leq p-1,$$

so that

$$K_{n,k}^{1/Z(n,0)} = \left(\left[\frac{K_{n,k}}{K_{n,k-1}} \right]^{1/Z(n,k)} \right)^{\frac{Z(n,k)}{Z(n,0)}} K_{n,k-1}^{1/Z(n,0)} \xrightarrow{n \rightarrow \infty} e^{\sum_{j=0}^k w_j}.$$

Finally, from Theorem 5.3, we have that

$$\lim_{n \rightarrow \infty} |P_{n,k}(z)|^{1/Z(n,0)} = e^{-U^{\mu_k}(z)}, \quad 0 \leq k \leq p-1, \quad (5.32)$$

locally uniformly on $\mathbb{C} \setminus [a_k, b_k]$. This last equality already proves (5.10) for the case $k = 0$. For $1 \leq k \leq p$, the corresponding result follows from (5.29)-(5.32).

5.5 Proof of Theorem 5.5

Since the coefficients a_n are positive, it follows from (3.24), (3.25), and (4.3) that for all $n \geq p$, $n \equiv k \pmod{p}$,

$$a_n = \frac{K_{n-p,k}^2}{K_{n,k}^2},$$

so that

$$\prod_{j=1}^m a_{pj+k} = \frac{K_{k,k}^2}{K_{mp+k,k}^2}, \quad 0 \leq k \leq p-1.$$

Then, using (5.31) and (2.48), we obtain

$$\lim_{m \rightarrow \infty} \left(\prod_{j=1}^m a_{pj+k} \right)^{1/m} = \lim_{m \rightarrow \infty} \left[(K_{k,k}^2 / K_{mp+k,k}^2)^{1/Z(mp+k,0)} \right]^{\frac{Z(mp+k,0)}{m}} = e^{-\frac{2p}{p+1} \sum_{j=0}^k w_j}.$$

5.6 Proof of Corollary 5.6

For each $0 \leq j \leq p$, let $f_j : \Gamma_{0,j} \rightarrow [a_0, b_0]$ be the function given by $f_j(t) = t^{p+1}$, which is clearly a homeomorphism. Since $Q_n(z) = z^\ell P_{n,0}(z^{p+1})$, Q_n has a zero at the origin of order ℓ , and its remaining zeros are the elements of the set $\{f_j^{-1}(\tau) : 0 \leq j \leq p, P_{n,0}(\tau) = 0\}$. Thus, for every continuous function F on $\Gamma_0 \cup \{0\}$, we have

$$\int F d\mu_{Q_n} = \frac{\ell F(0)}{n} + \frac{n-\ell}{n(p+1)} \sum_{j=0}^p \frac{1}{p+1} \sum_{P_{n,0}(\tau)=0} F(f_j^{-1}(\tau))$$

so that

$$\lim_{n \rightarrow \infty} \int F d\mu_{Q_n} = \sum_{j=0}^p \frac{1}{p+1} \int F(f_j^{-1}(\tau)) d\mu_0(\tau) = \sum_{j=0}^p \int F(t) d\tilde{\mu}_0|_{\Gamma_{0,j}}(t) = \int F d\tilde{\mu}_0.$$

5.7 Proof of Corollary 5.7

Since $\tilde{\mu}_k$ is rotationally symmetric and μ_k is the push forward of $\tilde{\mu}_k$ by the map $z \rightarrow z^{p+1}$, we have

$$\begin{aligned} U^{\mu_k}(z^{p+1}) &= \int_{a_k}^{b_k} \log \frac{1}{|z^{p+1} - \tau|} d\mu_k(\tau) = \int_{\Gamma_k} \log \frac{1}{|z^{p+1} - t^{p+1}|} d\tilde{\mu}_k(t) \\ &= \sum_{j=0}^p \int_{\Gamma_k} \log \frac{1}{|z - t\omega^j|} d\tilde{\mu}_k(t) = (p+1)U^{\tilde{\mu}_k}(z). \end{aligned} \quad (5.33)$$

Then, from (2.23), (5.10), and (2.48), we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} |\Psi_{n,k}(z)|^{1/n} &= \lim_{n \rightarrow \infty} \left(|\psi_{n,k}(z^{p+1})|^{1/Z(n,0)} \right)^{\frac{Z(n,0)}{n}} \\ &= e^{\frac{1}{p+1} (-U^{\mu_k}(z^{p+1}) + U^{\mu_{k-1}}(z^{p+1}) - 2 \sum_{j=1}^{k-1} w_j)} \\ &= e^{-U^{\tilde{\mu}_k}(z) + U^{\tilde{\mu}_{k-1}}(z) - \frac{2}{p+1} \sum_{j=1}^{k-1} w_j}, \end{aligned}$$

uniformly on compact subsets of $\mathbb{C} \setminus (\Gamma_k \cup \Gamma_{k-1} \cup \{0\})$, as desired.

6 Hermite-Padé approximation

6.1 Definitions and results

In this section we study the Hermite-Padé approximation to the system of functions

$$\widehat{s}_j(z) = \int_{\Gamma_0} \frac{ds_j(t)}{z-t}, \quad 0 \leq j \leq p-1, \quad (6.1)$$

where $(s_0, \dots, s_{p-1}) = \mathcal{N}(\sigma_0, \dots, \sigma_{p-1})$ is the Nikishin system of measures defined in (2.1). For this, we follow closely the method employed by Gonchar-Rakhmanov-Sorokin [7] in their study of Hermite-Padé approximants for generalized Nikishin systems on the real line.

The problem of Hermite-Padé approximation for the system of functions (6.1) is the following. Given a multi-index $\vec{n} = (n_0, n_1, \dots, n_{p-1}) \in \mathbb{Z}_+^p$, we seek a non-zero polynomial $Q_{\vec{n}}$ with $\deg(Q_{\vec{n}}) \leq |\vec{n}| := n_0 + \dots + n_{p-1}$ such that for every $j = 0, \dots, p-1$,

$$Q_{\vec{n}}(z) \widehat{s}_j(z) - Q_{\vec{n},j}(z) = O\left(\frac{1}{z^{n_j+1}}\right), \quad z \rightarrow \infty, \quad (6.2)$$

where $Q_{\vec{n},j}$ is the polynomial part in the Laurent series expansion of $Q_{\vec{n}} \widehat{s}_j$ at $z = \infty$. It is easy to see that such a polynomial $Q_{\vec{n}}$ exists, since the p conditions (6.2) can be expressed equivalently as a homogeneous system of $|\vec{n}|$ linear equations with $|\vec{n}| + 1$ unknowns (the coefficients of $Q_{\vec{n}}$), which always has a non-trivial solution. The vector of rational functions

$$\left(\frac{Q_{\vec{n},0}}{Q_{\vec{n}}}, \frac{Q_{\vec{n},1}}{Q_{\vec{n}}}, \dots, \frac{Q_{\vec{n},p-1}}{Q_{\vec{n}}} \right)$$

is called a Hermite-Padé approximant associated with \vec{n} for the system of functions (6.1). If we integrate the expression $z^l(Q_{\vec{n}}(z)\widehat{s}_j(z) - Q_{\vec{n},j}(z))$, $l = 0, \dots, n_j - 1$, along a closed contour that surrounds Γ_0 , it easily follows from (6.2) and (6.1) that the polynomial $Q_{\vec{n}}$ satisfies the multi-orthogonality conditions

$$\int_{\Gamma_0} Q_{\vec{n}}(z) z^l ds_j(z) = 0, \quad l = 0, \dots, n_j - 1, \quad 0 \leq j \leq p-1.$$

In this paper we will only consider Hermite-Padé approximants associated with multi-indices $\vec{n} = (n_0, \dots, n_{p-1}) \in \mathbb{Z}_+^p$ that are defined by

$$n_j = \left\lfloor \frac{n-j-1}{p} \right\rfloor + 1, \quad j = 0, \dots, p-1, \quad (6.3)$$

for a given integer $n \geq 0$. These multi-indices can be equivalently described as those satisfying the conditions $n_0 \geq n_1 \geq \dots \geq n_{p-1}$ and $n_{p-1} \geq n_0 - 1$, and are uniquely determined by their norm $|\vec{n}|$, which equals n is \vec{n} is defined by (6.3). Let I denote the sequence of such multi-indices.

Given $\vec{n} \in I$ with $|\vec{n}| = n$, we see that $Q_{\vec{n}}$ satisfies (2.13). Thus, if we assume, as we will do, that $Q_{\vec{n}}$ is monic, then by the normality of the Nikishin system we have that $Q_{\vec{n}}$ is unique and is given by $Q_{\vec{n}} = Q_n$. Moreover, since

$$\int_{\Gamma_0} \frac{Q_n(t)}{z-t} ds_j(t) = O\left(\frac{1}{z^{n_j+1}}\right), \quad z \rightarrow \infty,$$

from (6.2) and the identity

$$Q_n(z) \widehat{s}_j(z) - \int_{\Gamma_0} \frac{Q_n(z) - Q_n(t)}{z-t} ds_j(t) = \int_{\Gamma_0} \frac{Q_n(t)}{z-t} ds_j(t) \quad (6.4)$$

it follows that the polynomials

$$Q_{n,j}(z) := \int_{\Gamma_0} \frac{Q_n(z) - Q_n(t)}{z-t} ds_j(t), \quad j = 0, \dots, p-1, \quad (6.5)$$

are the numerators of the Hermite-Padé approximant associated with \vec{n} .

Definition 6.1. Given $n \geq 0$, for each $j = 0, \dots, p-1$ we define

$$\Phi_{n,j+1}(z) = \int \frac{Q_n(t)}{z-t} ds_j(t), \quad (6.6)$$

$$\delta_{n,j}(z) = \frac{\Phi_{n,j+1}(z)}{Q_n(z)}. \quad (6.7)$$

From (6.4)-(6.7) we deduce that

$$\delta_{n,j} = \widehat{s}_j - \frac{Q_{n,j}}{Q_n}, \quad j = 0, \dots, p-1, \quad (6.8)$$

i.e., $\delta_{n,j}$ is the remainder in the approximation of \widehat{s}_j by the j th component of the n th Hermite-Padé approximant.

Theorem 6.2. *Under the same assumptions of Theorem 5.3, we have that for every $j = 0, \dots, p-1$,*

$$\lim_{n \rightarrow \infty} |\delta_{n,j}(z)|^{1/n} = e^{-U^{\tilde{\mu}_1}(z) + 2U^{\tilde{\mu}_0}(z) - \frac{2}{p+1}w_0} \quad (6.9)$$

uniformly on compact subsets of $\mathbb{C} \setminus (\bigcup_{i=0}^{j+1} \Gamma_i \cup \{0\})$, where the measures $\tilde{\mu}_0, \tilde{\mu}_1$ are defined in (5.11), and $w_0 = \bar{w}_0$ is the equilibrium constant in (5.19). In particular, for every $j = 0, \dots, p-1$,

$$\lim_{n \rightarrow \infty} \frac{Q_{n,j}(z)}{Q_n(z)} = \widehat{s}_j(z), \quad (6.10)$$

uniformly on compact subsets of $\mathbb{C} \setminus (\bigcup_{i=0}^{j+1} \Gamma_i \cup \{0\})$.

Before we give the proof of Theorem 6.2, we make some remarks and prove an auxiliary result.

Note that $\Phi_{n,1} = \Psi_{n,1}$. More generally, one can show (see e.g. [7, pg. 691]) that for every $k = 1, \dots, p$,

$$\Phi_{n,k}(z) = \sum_{i=1}^k (-1)^{i-1} \widehat{s}_{i,k-1}(z) \Psi_{n,i}(z), \quad z \in \mathbb{C} \setminus \bigcup_{l=0}^{k-1} \Gamma_l, \quad (6.11)$$

where $\widehat{s}_{i,k-1}(z)$ denotes the Cauchy transform of the measure $s_{i,k-1} = \langle \sigma_i, \dots, \sigma_{k-1} \rangle$ (cf. (2.3)), and we understand $\widehat{s}_{k,k-1}(z) \equiv 1$. Observe that (2.11) implies that the function $\widehat{s}_{i,k-1}(z)$ does not vanish on $\mathbb{C} \setminus (\Gamma_i \cup \{0\})$.

Lemma 6.3. *Let $(\mu_0, \dots, \mu_{p-1})$ be the vector equilibrium measure satisfying (5.9), and w_0, \dots, w_{p-1} the associated equilibrium constants. Then, for every $k = 0, \dots, p-1$ we have*

$$2U^{\mu_k}(z) - U^{\mu_{k-1}}(z) - U^{\mu_{k+1}}(z) - 2w_k < 0, \quad \text{for all } z \in \overline{\mathbb{C}} \setminus \text{supp}(\mu_k), \quad (6.12)$$

where $U^{\mu_{-1}} \equiv 0$, $U^{\mu_p} \equiv 0$.

Proof. According to (5.19) and (5.28), we have

$$2U^{\mu_k}(x) - U^{\mu_{k-1}}(x) - U^{\mu_{k+1}}(x) - 2w_k = 0, \quad x \in \text{supp}(\mu_k), \quad k = 0, \dots, p-1. \quad (6.13)$$

This implies that U^{μ_k} is continuous on $\text{supp}(\mu_k)$, and hence U^{μ_k} is continuous on \mathbb{C} for all k . The measure $2\mu_k - \mu_{k-1} - \mu_{k+1}$ has total mass $1 + \frac{1}{p}$ if $k = 0$ and has total mass 0 for all other values of k . Therefore the function $2U^{\mu_k}(z) - U^{\mu_{k-1}}(z) - U^{\mu_{k+1}}(z) - 2w_k$ is subharmonic on $\overline{\mathbb{C}} \setminus \text{supp}(\mu_k)$. By the maximum principle for subharmonic functions applied to this function, (6.13) implies (6.12). \square

6.2 Proof of Theorem 6.2

We already observed that $\Phi_{n,1} = \Psi_{n,1}$. In virtue of (5.12), (5.13) and (6.7), we get

$$\lim_{n \rightarrow \infty} |\delta_{n,0}(z)|^{1/n} = \lim_{n \rightarrow \infty} \frac{|\Psi_{n,1}(z)|^{1/n}}{|Q_n(z)|^{1/n}} = e^{-U^{\tilde{\mu}_1}(z) + 2U^{\tilde{\mu}_0}(z) - \frac{2}{p+1}w_0},$$

uniformly on compact subsets of $\mathbb{C} \setminus (\Gamma_1 \cup \Gamma_0 \cup \{0\})$, which is (6.9) for $j = 0$.

Let $1 \leq j \leq p-1$. By (6.11) we can write for $z \in \mathbb{C} \setminus \left(\bigcup_{l=0}^j \Gamma_l \cup \{0\}\right)$,

$$\Phi_{n,j+1}(z) = \widehat{s}_{1,j}(z)\Psi_{n,1}(z) \left(1 - \frac{\widehat{s}_{2,j}(z)\Psi_{n,2}(z)}{\widehat{s}_{1,j}(z)\Psi_{n,1}(z)} + \dots + (-1)^j \frac{\Psi_{n,j+1}(z)}{\widehat{s}_{1,j}(z)\Psi_{n,1}(z)}\right). \quad (6.14)$$

Now, applying (5.33) and (6.12), we see that for every $i = 1, \dots, j$,

$$-U^{\tilde{\mu}_i}(z) + U^{\tilde{\mu}_{i-1}}(z) - \frac{2}{p+1} \sum_{l=0}^{i-1} w_l > -U^{\tilde{\mu}_{i+1}}(z) + U^{\tilde{\mu}_i}(z) - \frac{2}{p+1} \sum_{l=0}^i w_l,$$

for all $z \in \mathbb{C} \setminus \text{supp}(\tilde{\mu}_i)$. This implies, in virtue of (5.13), that

$$\lim_{n \rightarrow \infty} \frac{\Psi_{n,i+1}(z)}{\Psi_{n,1}(z)} = 0, \quad 1 \leq i \leq j,$$

locally uniformly on $\mathbb{C} \setminus \left(\bigcup_{l=0}^{j+1} \Gamma_l \cup \{0\}\right)$. This and (6.14) give

$$\lim_{n \rightarrow \infty} |\delta_{n,j}(z)|^{1/n} = \lim_{n \rightarrow \infty} \frac{|\Phi_{n,j+1}(z)|^{1/n}}{|Q_n(z)|^{1/n}} = \lim_{n \rightarrow \infty} \frac{|\Psi_{n,1}(z)|^{1/n}}{|Q_n(z)|^{1/n}} = e^{-U^{\tilde{\mu}_1}(z) + 2U^{\tilde{\mu}_0}(z) - \frac{2}{p+1}w_0},$$

uniformly on compact subsets of $\mathbb{C} \setminus \left(\bigcup_{l=0}^{j+1} \Gamma_l \cup \{0\}\right)$. From (6.13) for $k = 0$ and (5.33) we see that $-U^{\tilde{\mu}_1}(z) + 2U^{\tilde{\mu}_0}(z) - \frac{2}{p+1}w_0 < 0$ on $\mathbb{C} \setminus \left(\bigcup_{l=0}^{j+1} \Gamma_l \cup \{0\}\right)$, hence (6.10) follows from (6.7) and (6.9).

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